Unit 1: Relationships Between Quantities

In this unit you will study quantitative relationships. You will learn how important units are for interpreting problems and setting up equations. The focus will be on both one- and two-variable linear and exponential equations. There will also be examples of modeling with inequalities.

KEY STANDARDS

Reason quantitatively and use units to solve problems.

MCC9-12.N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★

MCC9-12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.★

MCC9-12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Interpret the structure of expressions.

MCC9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.★ (*Emphasis on linear expressions and exponential expressions with integer exponents*.)

MCC9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients.★ (*Emphasis on linear expressions and exponential expressions with integer exponents*.)

MCC9-12.A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.★ (*Emphasis on linear expressions and exponential expressions with integer exponents*.)

Create equations that describe numbers or relationships.

MCC9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear ~~and quadratic functions, and simple rational~~ and exponential functions.★

MCC9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★ (*Limit to linear and exponential equations and, in the case of exponential equations, limit to situations requiring evaluation of exponential functions at integer inputs*.)

MCC9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*. ★ (*Limit to linear equations and inequalities*.)

MCC9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★ (*Limit to formulas with a linear focus.*)

QUANTITIES AND UNITS

KEY IDEAS

1. A ***quantity*** is an exact amount or measurement. One type of quantity is a simple count, such as 5 eggs or 12 months. A second type of quantity is a measurement, which is an amount of a specific unit. Examples are 6 feet and 3 pounds.

2. A quantity can be exact or approximate. When an approximate quantity is used, it is important that we consider its level of accuracy. When working with measurements, we need to determine what level of accuracy is necessary and practical. For example, a dosage of medicine would need to be very precise. An example of a measurement that does not need to be very precise is the distance from your house to a local mall. The use of an appropriate unit for measurements is also important. For example, if you want to calculate the diameter of the Sun, you would want to choose a very large unit as your measure of length, such as miles or kilometers. Conversion of units can require approximations.

   Example:

   Convert 5 miles to feet.

   Solution:

   We know 1 mile is 5280 feet.

$$5 \text{ miles} \times \frac{5280 \text{ feet}}{1 \text{ mile}} = 26,400 \text{ feet}$$

   We can approximate that 5 miles is close to 26,000 feet.

3. The context of a problem tells us what types of units are involved. ***Dimensional analysis*** is a way to determine relationships among quantities using their dimensions, units, or unit equivalencies. Dimensional analysis suggests which quantities should be used for computation in order to obtain the desired result.

Example:

The number of calories a person burns doing an activity can be approximated using the formula $C = kmt$, where m is the person's weight in pounds and t is the duration of the activity in minutes. Find the units for the coefficient k.

Solution:

The coefficient k is a rate of burning calories specific to the activity. We can determine the units of k through dimensional analysis. We include the units for the quantities that we know, and then use them to determine what the units for the quantity represented by k are.

$$C \text{ calories} = k \times m \text{ pounds} \times t \text{ minutes}$$

The coefficient k is a combination of other units. In this example, the coefficient k would be measured in calories per pound-minute. We can see that only a quantity with the unit calories per pound-minute can produce a quantity with calories as its unit when multiplied by quantities with pounds and minutes as their units.

4. The process of dimensional analysis is also used to convert from one unit to another. Knowing the relationship between units is essential for unit conversion.

Example:

Convert 60 miles per hour to feet per minute.

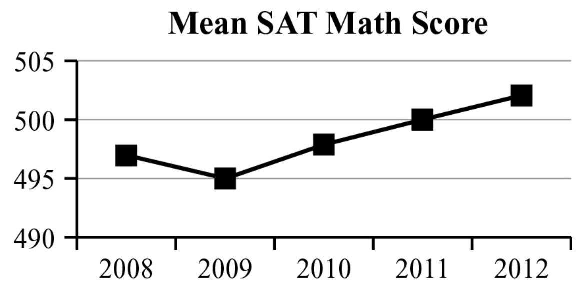
Solution:

To convert the given units, we use a form of dimensional analysis. We will multiply 60 mph by a series of ratios where the numerator and denominator are in different units but equivalent to each other. The ratios are carefully chosen to introduce the desired units.

$$\frac{60 \text{ miles}}{\text{hr}} \times \frac{1 \text{hr}}{60 \text{ minutes}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} = \frac{60 \times 5280 \text{ feet}}{60 \times 1 \text{ minute}} = 5280 \text{ feet per minute}$$

5. When data are displayed in a graph, the units and scale features are keys to interpreting the data. Breaks or an abbreviated scale in a graph should be noted as they can cause a misinterpretation of the data.

The sample graph on the next page shows the mean mathematics SAT scores for a school over a five-year period. All of the scores are between 200 and 800. When the vertical scale does not start at 200, it can exaggerate the change in the mean scores from year to year. The chart makes the changes in the mean mathematics SAT scores over the past five years look significant. In reality, the changes were very modest. They ranged only between 495 and 502 over those years.



6. The measurements we use are often approximations. It is routinely necessary to determine reasonable approximations.

Example:

When Justin goes to work, he drives at an average speed of 65 miles per hour. It takes about 1 hour and 30 minutes for Justin to arrive at work. His car travels about 25 miles per gallon of gas. If gas costs \$3.65 per gallon, how much money does Justin spend to travel each mile to work?

Solution:

Calculate the distance traveled at 65 miles per hour for 1.2 hours.

$$65 \text{ miles per hour} \bullet 1.5 \text{ hour} = 97.5 \text{ miles}$$

Justin can get 25 miles from 1 gallon of gas, and 100 miles from 4 gallons of gas, which is a little more than 97.5 miles. So, he'll need about 4 gallons of gas.

Justin pays \$3.65 per gallon of gas and it gets him 25 miles on the highway. So, it'll cost him $\frac{\$3.65}{25 \text{ miles}} \approx \frac{\$0.15}{1 \text{ mile}}$ or 15 cents for each mile he travels on the highway.

Important Tips

- Whenever possible, include the unit or the items being counted when referring to a quantity.

- The use of appropriate units for various measurements is very important. There are different systems of measurement for the same dimensions, such as distance and weight. Within a system there are units of different sizes. It is essential to know the relative size of units within the same system and be able to convert among them. Rounding or limiting the number of digits used in unit conversions is often advisable.

- There are situations when the units in an answer tell us if the answer is wrong. For example, if the question called for weight and the answer is given in cubic feet, we know the answer cannot be correct.

REVIEW EXAMPLES

1) The formula for density d is $d = \frac{m}{v}$, where m is mass and v is volume. If mass is measured in kilograms and volume is measured in cubic meters, what is the unit rate for density?

Solution:

The unit rate for density is $\frac{\text{kilograms}}{\text{meters}^3}$, or $\frac{\text{kg}}{\text{m}^3}$.

2) A rectangular prism has a volume of 2 m^3, a length of 40 cm, and a width of 50 cm. What is the height of the prism?

Solution:

The volume of a rectangular prism is found by using the formula $V = l \times w \times h$, where V is volume, l is length, w is width, and h is height.

$$2\text{ m}^3 = 40\text{ cm} \times 50\text{ cm} \times h$$
$$2\text{ m}^3 = 200\text{ cm}^2 \times h$$

To find the height, h, the units on both sides must be the same. So, convert the volume from meters to centimeters. One meter is equivalent to 100 centimeters. Perform the conversion by multiplying 2 m^3 by ratios equivalent to 1. Each ratio should contain 100 centimeters in the numerator and 1 meter in the denominator.

$$2\,\text{m} \times \text{m} \times \text{m} \times \frac{100\,\text{cm}}{1\,\text{m}} \times \frac{100\,\text{cm}}{1\,\text{m}} \times \frac{100\,\text{cm}}{1\,\text{m}}$$

Cancel the units where possible and multiply the remaining factors. The product is the converted measurement.

$$2\,\text{m} \times \text{m} \times \text{m} \times \frac{100\,\text{cm}}{1\,\text{m}} \times \frac{100\,\text{cm}}{1\,\text{m}} \times \frac{100\,\text{cm}}{1\,\text{m}} = 20{,}000\,\text{cm} \times \text{cm} \times \text{cm} = 20{,}000\,\text{cm}^3$$

Use the converted measurement in the volume formula to solve for h.

$$20{,}000\text{cm}^3 = 200\text{ cm}^2 \times h$$
$$\frac{20{,}000\,\text{cm}^3}{200\,\text{cm}^2} = \frac{200\,\text{cm}^2 \times h}{200\,\text{cm}^2}$$

$$h = 100\text{ cm, or } h = 1\text{ meter}$$

EOCT Practice Items

1) A rectangle has an area of 12 m^2 and a width of 400 cm. What is the length of the rectangle?

A. 3 cm
B. 30 cm
C. 300 cm
D. 3000 cm

[Key: C]

2) What is the area of a circle with a circumference of 43.98226 inches? (Use 3.14159 for π.)

A. 153.44029 in.^2

B. 153.93791 in.^2

C. 153.9325 in.^2

D. 153.9394 in.^2

[Key: B]

3) The tension caused by a wave moving along a string is found using the formula $T = \frac{mv^2}{L}$. If *m* is the mass of the string in grams, *L* is the length of the string in centimeters, and *v* is the velocity of the wave in centimeters per second, what is the unit of the tension of the string, *T*?

A. gram-centimeters per second squared

B. centimeters per second squared

C. grams per centimeter-second squared

D. centimeters squared per second

[Key: A]

STRUCTURE OF EXPRESSIONS

KEY IDEAS

1. ***Arithmetic expressions*** are comprised of numbers and operation signs.

   Examples: $2 + 4$ and $4^2 - (10 + 3)$

   Algebraic expressions contain one or more variables.

   Examples: $2x + 4$, $4x^2 - (10 + 3x)$, and $\dfrac{\sqrt{9} + 2t}{5}$

   The parts of expressions that are separated by addition or subtraction signs are called ***terms***. Terms are usually composed of numerical factors and variable factors. The numerical factor is called the ***coefficient***.

   Consider the algebraic expression $4x^2 + 7xy - 3$. It has three terms: $4x^2$, $7xy$, and 3. For $4x^2$, the coefficient is 4 and the variable factor is x. For $7xy$, the coefficient is 7 and the variable factors are x and y. The third term, 3, has no variables and is called a ***constant***.

2. To interpret a formula, it is important to know what each variable represents and to understand the relationships between the variables.

   An example would be th e compound interest formula $A = P(1+r)^t$, where A is the amount accumulated or balance, P is the principal or starting amount deposited, r is the interest rate, and t is the number of years.

   This formula is used to calculate the balance of an account when the annual interest is compounded. The formula shows the product of the starting amount P and a power with the expression within the parentheses as its base. The value r is usually given as a percent. When r is converted to a decimal and added to 1, the result becomes the base of the power t and is used to determine how much of the principal will grow in interest. The starting amount P is multiplied by the growth power to determine how much money will accumulate. The formula is essentially a growth formula. All growth formulas include a growth power.

   Another example would be the formula used to estimate the number of calories burned while jogging, $C = 0.075mt$, where m represents a person's body weight in pounds and t is the number of minutes spent jogging. This formula tells us that the number of calories burned depends on a person's body weight and how much time is spent jogging. The coefficient, 0.075, is the factor used for jogging. In calculating the number of calories burned, the coefficient differs from activity to activity.

Important Tips

- Look for powers. They are important in recognizing how the variables relate to one another.
- Consider how a coefficient affects a term. Try different coefficients for the same term and explore the effects.

REVIEW EXAMPLES

1) An amount of $1000 is deposited into a bank account that pays 4% annual interest. What would be the bank account balance after 5 years?

Solution:

Use the formula $A = P(1 + r)^t$. P is $1000, r is 4% or 0.04, and t is 5 years.

$$A = 1000(1 + 0.04)^5 = 1000 \times 1.216652902 \approx 1,216.65$$

The balance after 5 years would be $1,216.53.

2) The number of calories burned during exercise depends on the activity. The formulas for two activities are given.

$$C_1 = 0.012mt \text{ and } C_2 = 0.032mt$$

a. If one activity is cooking and the other is bicycling, identify the formula that represents each activity. Explain your answer.

b. What value would you expect the coefficient to have if the activity were reading? Include units and explain your answer.

Solution:

a. The coefficient of the variable term *mt* tells us how strenuous the activity is. Since bicycling is more strenuous than cooking, its formula would have a higher coefficient. Therefore, the formula for bicycling is likely $C_1 = 0.032mt$.

b. Since reading is less strenuous than cooking, the number of calories burned is probably fewer, and the coefficient is probably smaller. Expect a coefficient smaller than 0.012 calories per pound-minute for reading.

EOCT Practice Items

1) The kinetic energy of an object in motion is found using the formula $KE = \frac{1}{2}mv^2$, where *m* is the mass of the object in kilograms and *v* is the velocity of the object in meters per second. If the velocity of the object is 20 meters per second, what is the coefficient of *m*?

A. 10 meters/second

B. 20 meters/second

C. 200 meters squared/second squared

D. 400 meters squared/second squared

[Key: C]

2) A certain population of bacteria has a growth rate of 0.02 bacteria/hour. The formula for the growth of the bacteria's population is $A = P_0(2.71828)^{0.02t}$, where P_0 is the original population and *t* is the time in hours.

If you begin with 200 bacteria, approximately how many of the bacteria can you expect after 100 hours?

A. 7.38905

B. 271.828

C. 1477.81

D. 20,000

[Key: C]

EQUATIONS AND INEQUALITIES

KEY IDEAS

1. Problems in quantitative relationships generally call for us to determine how the quantities are related to each other, use a variable or variables to represent unknowns, and solve some equation or inequality. There are problems we can model using one unknown in an equation. We may want to find a missing angle in a triangle or the number of dimes in a collection of coins.

   Example:

   Two angles of a triangle measure 30° and 70°. What is the measure of the third angle?

   Solution:

   The sum of the angle measures in a triangle are 180°. Let $x°$ represent the measure of the third angle.

   $30° + 70° + x° = 180°$

   Next, we solve for x.

$30 + 70 + x = 180$	Write the original equation.
$100 + x = 180$	Combine like terms.
$x = 80$	Subtract 100 from both sides.

   The third angle measures 80°.

2. There are also problems that can be modeled with inequalities. We may want to determine the number of weeks we need to save money in order to buy something, or the price to charge for a product or service.

   Example:

   A social media website currently has only 1000 members. The number of people that join the website triples every 6 months. How long will it take for the website to have at least 1,000,000 members?

   Solution:

   Tripling means to multiply by 3. We need to determine the number of factors of 3 that should be multiplied by 1000 so that the product is more than 1,000,000 ($1{,}000 \times 3 \times 3 \times \ldots > 1{,}000{,}000$). The solution can be set up this way:

$$1000 \times 3^x > 1{,}000{,}000$$

One way to find the solution is to divide 1,000,000 by 1000 first. The result is $3^x > 1000$.

Next, we need to determine the number of times we must multiply 3 by itself so that the product is greater than 1000. We can make a chart.

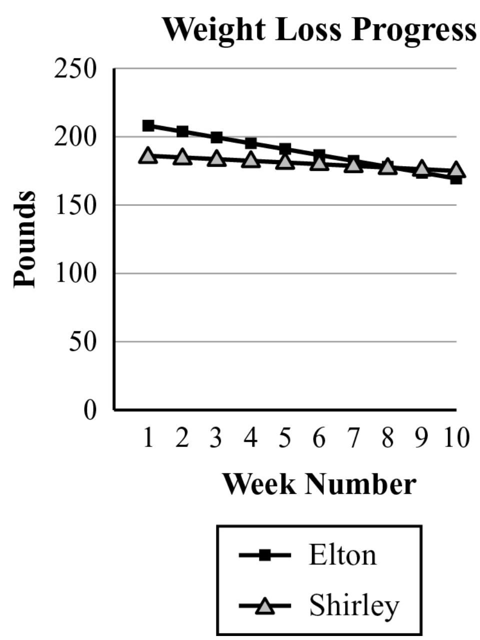
3^1	3^2	3^3	3^4	3^5	3^6	3^7
3	9	27	81	243	729	2187

The answer is 7 times. That means it will take 3.5 years because the number of members triples every six months, not each year.

3. Some situations involve a pair of related variables. When there are two variables, there must be two equations or inequalities. We can solve many of these types of problems by graphing.

Example:

Elton loses 5 pounds each week. He started at 218 pounds on Week 1. Shirley was 186 pounds on Week 1 and she loses 1 pound each week. The graph shows Elton's and Shirley's weight by week.



a. What are the equations for Elton's and Shirley's weight loss?
b. How much do both Elton and Shirley weigh at Week 8?

Solution:

a. Elton's weight loss is represented by the equation $w = 218 - 5t$, where w is his weight and t is the week number. Shirley's weight loss is represented by the equation $w = 180 - 1t$.
b. At Week 8, Elton weighs $218 - 5 \times 8 = 178$ pounds. That same week, Shirley would weigh $186 - 1 \times 8 = 178$ pounds. The graph shows us that at Week 8 both Elton and Shirley weigh the same.

4. There are situations with two related variables that we can model with inequalities. These inequalities are called ***constraints*** and can be graphed in a coordinate plane. The intersections of the boundaries of the graphs of the constraints form the region where the solutions to the problem lie.

Example:

Mark only has $14 for lunch. He wants to have at least one sandwich and one drink. If the deli sells sandwiches for $5 and drinks for $2, what combinations could Mark have for lunch?

Solution:

We could make a table of possibilities.

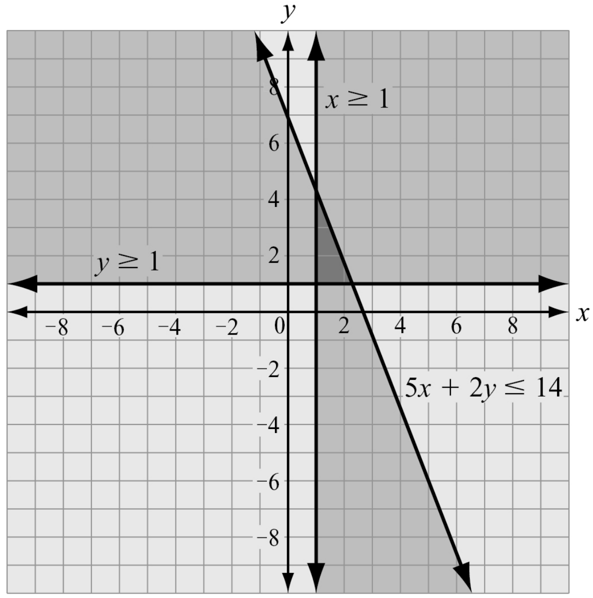
Sandwiches	Drinks
1	1, 2, 3, or 4
2	1 or 2

If we let x be the number of sandwiches and y be the number of drinks, we could write a system of inequalities:

$$\begin{Bmatrix} 5x + 2y \le 14 \\ x \ge 1 \\ y \ge 1 \end{Bmatrix}$$

The graph of the system above shows three regions called half-planes, and the possible combinations of sandwiches and drinks are the coordinates of the points where all three regions overlap. The graph has three boundary lines, one for each inequality. The shading is below the boundary for $5x + 2y \le 14$, to the right of $x \ge 1$, and above $y \ge 1$. The triangular region where all the shaded regions overlap contains possible numbers of sandwiches and drinks that Mark chose for lunch.

Mark's Meal Possibilities



The inequalities here show the constraints on the problem: Mark only has \$14 to spend and he needs to have at least one sandwich and a drink. The whole-numbered coordinates in the triangular region are given by these points: (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), and (2, 2). The first number in each ordered pair tells us the number of sandwiches and the second number tells us the number of drinks.

5. In some cases a number is not required for a solution. Instead, we want to know how a certain variable relates to another.

Example:

Solve the equation $v = gt^2$ for g.

Solution:

To solve for g we need to divide both sides of the equation by t^2. However, this division is not allowed if t equals zero. So, we must restrict the values of t when solving for g.

$v = gt^2$	Write the original equation.
$\frac{v}{t^2} = \frac{gt^2}{t^2}$	Divide each side by t^2.
$\frac{v}{t^2} = g,\ t \neq 0$	Simplify. Restrict the values of t.

REVIEW EXAMPLES

1) The city of Arachna has a spider population that has been doubling every year. If there are about 100,000 spiders this year, how many will there be 4 years from now?

Solution:

We know that doubling means multiplying by 2. We can use this equation:

$$S = 100{,}000 \times 2 \times 2 \times 2 \times 2 \text{ or } 100{,}000 \times 2^4$$

$$S = 1{,}600{,}000$$

There will be 1,600,000 spiders 4 years from now.

2) The Jones family has twice as many tomato plants as pepper plants. If there are 21 plants in their garden, how many plants are pepper plants?

Solution:

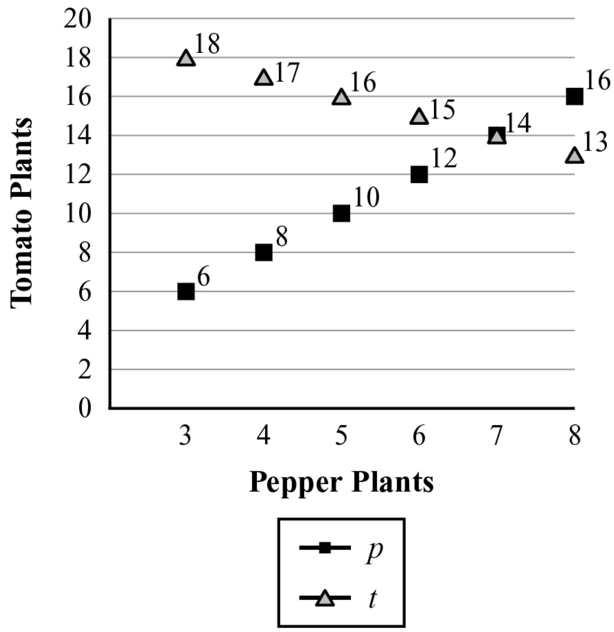
If we let p be the number of pepper plants, and t be the number of tomato plants, we can model the situation with these two equations:

$$t = 2p$$
$$p + t = 21$$

Now we can create tables of values for each equation, and then graph the values. We should be sure to use the same values of t for both tables. Solve $p + t = 21$ for t to make finding values simpler.

$t = 2p$	
p	***t***
3	6
4	8
5	10
6	12
7	14
8	16

$t = 21 - p$	
p	***t***
3	18
4	17
5	16
6	15
7	14
8	13



From the graph we see the equations share a common point (7, 14). That means there are 7 pepper plants and 14 tomato plants.

3) The angles of a triangle measure $x°$, $2x°$, and $6x°$. Solve for x.

Solution:

Use the equation $x + 2x + 6x = 180$ to represent the relationship.

Combine like terms: $9x = 180$.

Divide each side by 9: $\frac{9x}{9} = \frac{180}{9}$; $x = 20$

4) A business invests $6,000 in equipment to produce a product. Each unit of the product costs $0.90 to produce and is sold for $1.50. How many units of the product must be sold in order for the business to make a profit?

Solution:

Let C be the total cost of producing x units. Represent the total cost with an equation.

$$C = 0.90x + 6000$$

Let R be the total revenue from selling x units. Represent the total revenue with an equation.

$$R = 1.50x$$

A break-even point is reached when the total revenue R equals the total cost C. A profit occurs when revenue is greater than cost.

$$1.50x > 0.90x + 6000$$

Solve the inequality.

Subtract $0.90x$ from each side: $0.60x > 6000$

Divide each side by 0.60: $x > 10{,}000$

More than 10,000 units must be sold in order for the business to make a profit.

EOCT Practice Items

1) Two angles of a triangle measure 20° and 50°. What is the measure of the third angle?

A. 30°

B. 70°

C. 110°

D. 160°

[Key: C]

2) What is the solution to the equation $P = 2l + 2w$ when solved for w?

A. $w = \frac{2l}{P}$

B. $w = \frac{2l - P}{2}$

C. $w = 2l - \frac{P}{2}$

D. $w = \frac{P - 2l}{2}$

[Key: D]

3) Bruce owns a business that produces widgets. He must bring in more in revenue than he pays out in costs in order to turn a profit.

- **It costs $10 in labor and materials to make each of his widgets.**
- **His rent each month for his factory is $4000.**
- **He sells each widget for $25**.

How many widgets does Bruce need to sell each month to make the minimum profit?

A. 160

B. 260

C. 267

D. 400

[Key: C]