Unit 3: Linear and Exponential Functions

In Unit 3, students will learn function notation and develop the concepts of domain and range. They will discover that functions can be combined in ways similar to quantities, such as adding. Students will explore different ways of representing functions (e.g., graphs, rules, tables, sequences) and interpret functions given graphically, numerically, symbolically, and verbally. Discovering how functions can be transformed, similar to shapes in geometry, and learning about how parameters affect functions are aspects of this unit. Students will also learn how to compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They will interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

KEY STANDARDS

Represent and solve equations and inequalities graphically

MCC9-12.A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). (*Focus on linear and exponential equations and be able to adapt and apply that learning to other types of equations in future courses*.)

MCC9-12.A.REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, ~~polynomial, rational, absolute value~~, exponential, ~~and logarithmic~~ functions.★

Understand the concept of a function and use function notation

MCC9-12.F.IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$. (*Draw examples from linear and exponential functions*.)

MCC9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. (*Draw examples from linear and exponential functions*.)

MCC9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$ *(n is greater than or equal to 1).* (*Draw connection to F.BF.2, which requires students to write arithmetic and geometric sequences*.)

Interpret functions that arise in applications in terms of the context

MCC9-12.F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and ~~periodicity~~.★ (*Focus on linear and exponential functions.*)

MCC9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function* h(n) *gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ (*Focus on linear and exponential functions.*)

MCC9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ (*Focus on linear functions and intervals for exponential functions whose domain is a subset of the integers*.)

Analyze functions using different representations

MCC9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ (*Focus on linear and exponential functions. Include comparisons of two functions presented algebraically*.)

MCC9-12.F.IF.7a Graph linear ~~and quadratic~~ functions and show intercepts, maxima, and minima.★

MCC9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude~~.★

MCC9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (*Focus on linear and exponential functions. Include comparisons of two functions presented algebraically*.)

Build a function that models a relationship between two quantities

MCC9-12.F.BF.1 Write a function that describes a relationship between two quantities.★ (*Limit to linear and exponential functions*.)

MCC9-12.F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. (*Limit to linear and exponential functions*.)

MCC9-12.F.BF.1b Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* (*Limit to linear and exponential functions*.)

MCC9-12.F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

Build new functions from existing functions

MCC9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (*Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept*.)

Construct and compare linear, ~~quadratic~~, and exponential models and solve problems

MCC9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.★

MCC9-12.F.LE.1a Prove that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals.★

MCC9-12.F.LE.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★

MCC9-12.F.LE.1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★

MCC9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

MCC9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, ~~quadratically, or (more generally) as a polynomial function~~.★

Interpret expressions for functions in terms of the situation they model

MCC9-12.F.LE.5 Interpret the parameters in a linear or exponential function in terms of a context. ★ (*Limit exponential functions to those of the form* $f(x) = b^x + k$.)

REPRESENT AND SOLVE EQUATIONS AND INEQUALITIES GRAPHICALLY

KEY IDEAS

1. The graph of a linear equation in two variables is a collection of ordered pair solutions in a coordinate plane. It is a graph of a straight line. Often tables of values are used to organize the ordered pairs.

Example:

Every year Silas buys fudge at the state fair. He buys peanut butter and chocolate. This year he intends to buy \$24 worth of fudge. If chocolate costs \$4 per pound and peanut butter costs \$3 per pound, what are the different combinations of fudge that he can purchase?

Solution:

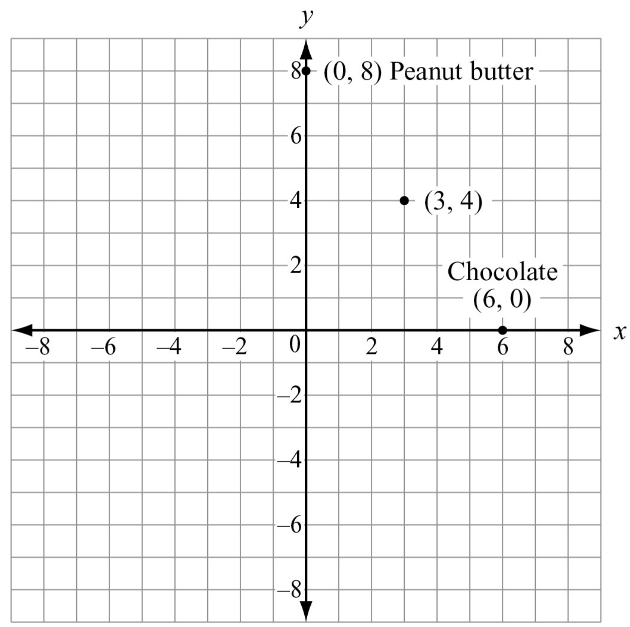
If we let x be the number of pounds of chocolate and y be the number pounds of peanut butter, we can use the equation $4x + 3y = 24$. Now we can solve this equation for y to make it easier to complete our table.

$4x + 3y = 24$	Write the original equation.
$4x - 4x + 3y = 24 - 4x$	Subtract $4x$ from each side.
$3y = 24 - 4x$	Simplify.
$\frac{3y}{3} = \frac{24-4x}{3}$	Divide each side by 3.
$y = \frac{24-4x}{3}$	Simplify.

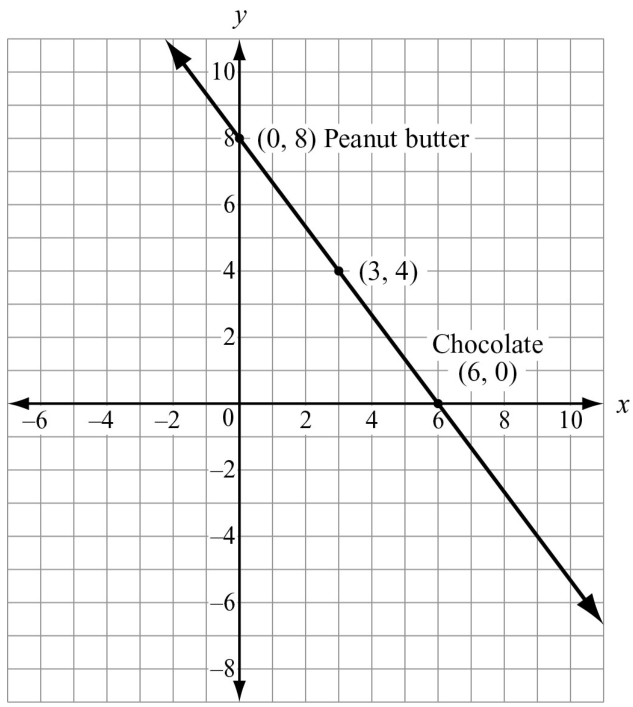
We will only use whole numbers in the table, because Silas will only buy whole pounds of the fudge.

Chocolate, ***x***	**Peanut butter,** ***y***
0	8
1	~~6⅔~~ (not a whole number)
2	~~5⅓~~ (not a whole number)
3	4
4	~~2⅔~~ (not a whole number)
5	~~1⅓~~ (not a whole number)
6	0

The ordered pairs from the table that we want to use are (0, 8), (3, 4), and (6, 0). The graph would look like the one shown below:



Based on the number of points in the graph, there are three possible ways that Silas can buy pounds of the fudges: 8 pounds of peanut butter only, 3 pounds of chocolate and 4 pounds of peanut butter, or 6 pounds of chocolate only. Notice that if the points on the graph were joined, they would form a line. If Silas allowed himself to buy partial pounds of fudge, then there would be many more possible combinations. Each combination would total $24 and be represented by a point on the line that contains (0, 8), (3, 4), and (6, 0).



2. We can also graph an inequality in two variables. The solution set of an inequality in two variables is a half-plane. The graph of an inequality is divided into an upper half-plane and a lower half-plane. If an inequality contains < or >, the boundary line is dashed and not included in the solution set. We call this type of inequality a ***strict*** inequality. If an inequality contains ≤ or ≥, the boundary line is solid and is included in the solution set.

Now we can revisit Silas and his fudge purchase.

Example:

Silas decides he does not have to spend exactly \$24 on the fudge, but he will not spend more than \$24. What are the different combinations of fudge purchases he can make?

Solution:

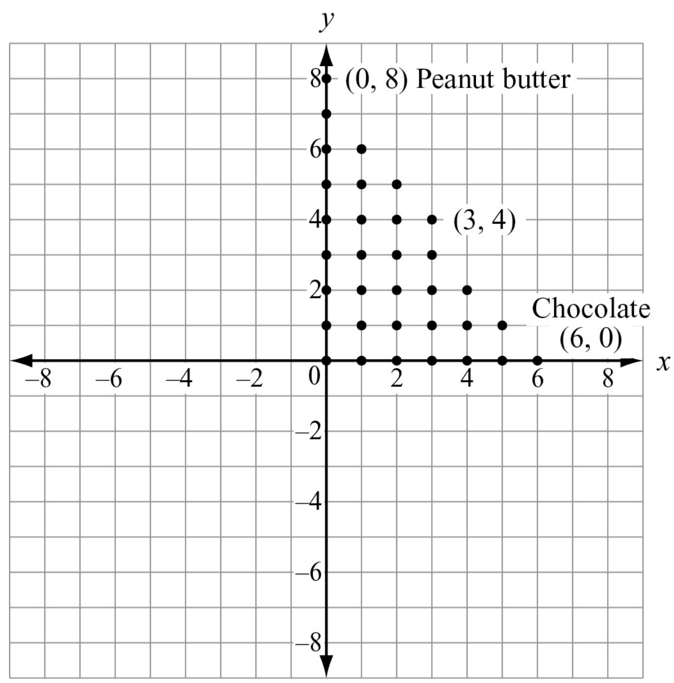
Our new relationship is the inequality $4x + 3y \leq 24$. We'll make a new table of values. We begin by making it easier to find the number of pounds of peanut butter by solving for y.

$4x+3y\leq 24$	Write the original equation.
$4x-4x+3y\leq 24-4x$	Subtract $4x$ from each side.
$3y\leq 24-4x$	Simplify.
$\frac{3y}{3}\leq\frac{24-4x}{3}$	Divide each side by 3.
$y\leq\frac{24-4x}{3}$	Simplify.

Once again, we will only use whole numbers from the table, because Silas will only buy whole pounds of the fudge.

Chocolate, x	**Peanut butter, $y \leq \dfrac{24-4x}{3}$**
0	0,1, 2, 3, 4, 5, 6, 7, 8
1	0, 1, 2, 3, 4, 5, 6, ~~6⅔~~
2	0, 1, 2, 3, 4, 5, ~~5⅓~~
3	0, 1, 2, 3, 4
4	0, 1, 2, ~~2⅔~~
5	0, 1, ~~1⅓~~
6	0

Each row in the table has at least one value that is a whole number in both columns. If Silas does not intend to spend all of his $24, there are many more combinations. Let's look at the graph.



From the points that lie on the graph and the rows in the table, there are 33 combinations of chocolate and peanut butter fudge that Silas can buy. The points on the graph do not look like a line. However, they all appear to fall below the line that would join (0, 8), (3, 4), and (6, 0). In fact, the line joining those three points would be the boundary line for the inequality that represents the combinations of pounds of fudge that Silas can purchase.

The points do not cover the entire half-plane below that line because it is impossible to purchase negative pounds of fudge. The fact that you cannot purchase negative amounts of fudge puts constraints, or limitations, on the possible x and y values. Both x and y values must be non-negative.

Important Tips

- Be careful in dealing with strict inequalities. Their boundary lines are dashed.
- When you multiply an inequality by a negative number, remember to reverse the inequality sign. The sign of a number or expression changes when it is multiplied or divided by a negative number. On a number line, negative and positive values are represented in opposite directions, so the inequality sign must be reversed to indicate that.
- When solving an inequality, remember to reverse the inequality sign if you need to multiply or divide by a negative number.
- Be sure that the order of the coordinates in an ordered pair of the form (x, y) is correct.

REVIEW EXAMPLES

1) Consider the equations $y = 2x - 3$ and $y = -x + 6$.

a. Complete the tables below, and then graph the equations on the same coordinate axes.

$\boldsymbol{y = 2x - 3}$	
x	y

$\boldsymbol{y = -x + 6}$	
x	y

b. Is there an ordered pair that satisfies both equations? If so, what is it?

c. Graph both equations on the same coordinate plane by plotting the ordered pairs from the tables and connecting the points.

d. Do the lines appear to intersect? If so, where? How can you tell that the point where the lines appear to intersect is a common point for both lines?

Solution:

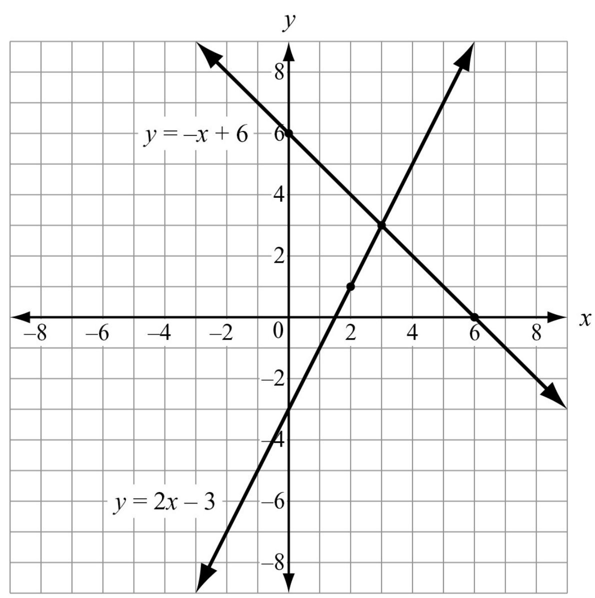
a.

$y = 2x - 3$	
x	y
–1	–5
0	–3
1	–1
2	1
3	3

$y = -x + 6$	
x	y
–1	7
0	6
1	5
2	4
3	3

b. Yes, the ordered pair (3, 3) satisfies both equations.

c.

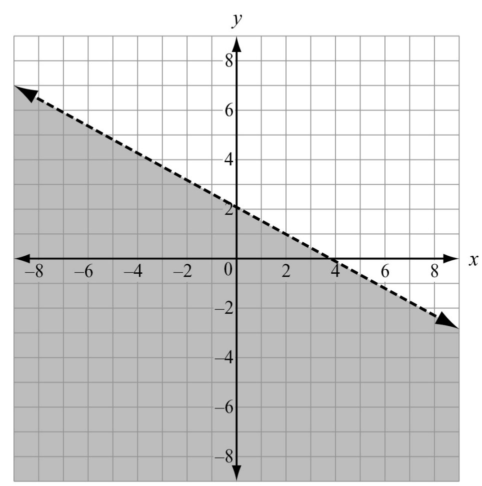


d. The lines appear to intersect at (3, 3). When $x = 3$ and $y = 3$ are substituted into each equation, the values satisfy both equations. This proves that (3, 3) lies on both lines, which means it is a common solution to both equations.

2) Graph the inequality $x + 2y < 4$.

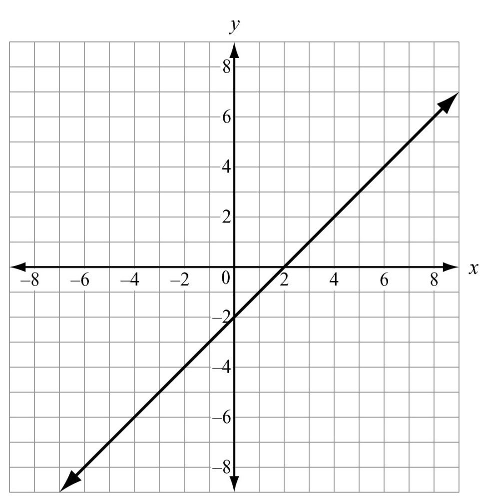
Solution:

The graph looks like a half-plane with a dashed boundary line. The shading is below the line because the points that satisfy the inequality fall strictly below the line.



EOCT Practice Items

1) Which equation corresponds to the graph shown?



A. $y = x + 1$

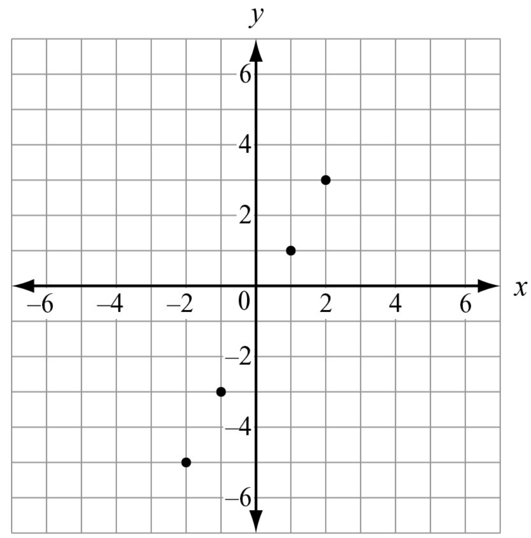
B. $y = 2x + 1$

C. $y = x – 2$

D. $y = 3x – 1$

[Key: C]

2) Which equation corresponds to the points in the coordinate plane?



A. $y = 2x - 1$
B. $y = x - 3$
C. $y = x + 1$
D. $y = x - 1$

[Key: A]

3) Based on the tables, what common point do the equations $y = -x + 5$ and $y = 2x - 1$ share?

$y = -x + 5$	
x	y
–1	6
0	5
1	4
2	3
3	2

$y = 2x - 1$	
x	y
–1	–3
0	–1
1	1
2	3
3	5

A. (1, 1)
B. (3, 5)
C. (2, 3)
D. (3, 2)

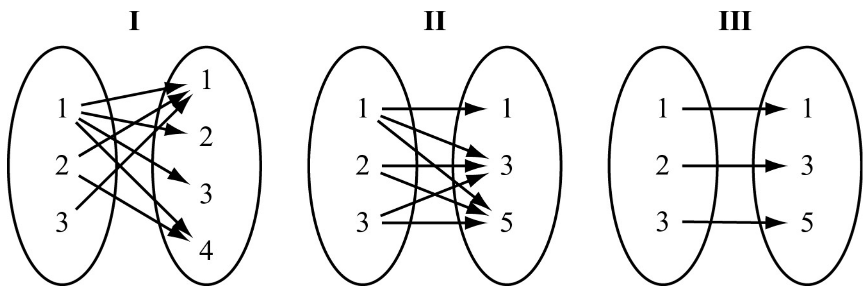
[Key: C]

UNDERSTAND THE CONCEPT OF A FUNCTION AND USE FUNCTION NOTATION

KEY IDEAS

1. There are many ways to show how pairs of quantities are related. Some of them are shown below.

- ***Mapping Diagrams***



- ***Sets of Ordered Pairs***

Set I: {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 4), (3, 1)}

Set II: {(1, 1), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)}

Set III: {(1, 1), (2, 3), (3, 5)}

- ***Tables of Values***

I

x	*y*
1	1
1	2
1	3
1	4
2	1
2	4
3	1

II

x	*y*
1	1
1	3
1	5
2	3
2	5
3	3
3	5

III

x	*y*
1	1
2	3
3	5

The relationship shown in Mapping Diagram I, Set I, and Table I all represent the same paired numbers. Likewise, Mapping II, Set II, and Table II all represent the same quantities. The same goes for the third groups of displays.

Notice the arrows in the mapping diagrams are all arranged from left to right. The numbers on the left side of the mapping diagrams are the same as the x-coordinates in the ordered pairs as well as the values in the first column of the tables. Those numbers are called the input values of a quantitative relationship and are known as the ***domain***. The numbers on the right of the mapping diagrams, the y-coordinates in the ordered pair, and the values in the second column of the table are the output, or ***range***. Every number in the domain is assigned to at least one number of the range.

Mapping diagrams, ordered pairs, and tables of values are good to use when there are a limited number of input and output values. There are some instances when the domain has an infinite number of elements to be assigned. In those cases, it is better to use either an algebraic rule or a graph to show how pairs of values are related. Often we use equations as the algebraic rules for the relationships.

2. A ***function*** is a quantitative relationship wherein each member of the domain is assigned to exactly one member of the range. Of the relationships above, only III is a function. In I and II, there were members of the domain that were assigned to two elements of the range. In particular, in I, the value 1 of the domain was paired with 1, 2, 3, and 4 of the range.

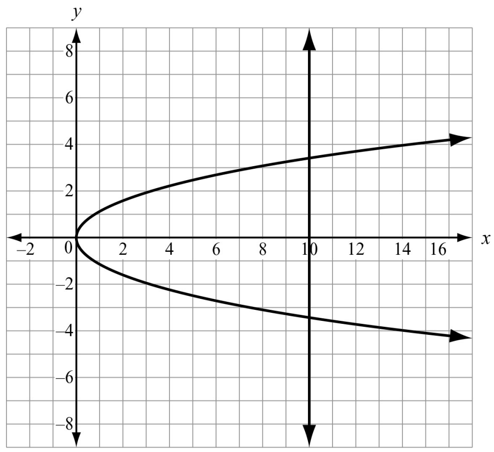
Consider the equation $y^2 = x$. Solving for y, you'll obtain the equation $y = \pm\sqrt{x}$. Therefore, when $x = 1$, then $y = 1$ or $y = -1$. That means the value 1 of the domain is paired with 1 and -1 of the range. Hence, the equation $y^2 = x$ is not a function.

You can also use the ***vertical line test*** to determine whether a relationship between pairs of values is a function. The vertical line test states that if a vertical line passes through more than one point on the graph of the relationship between two values, then it is not a function. If a vertical line passes through more than one point, then there is more than one value in the range that corresponds to one value in the domain.

Example:

Graph $y^2 = x$ to determine if it is a function.

Solution:



Because the vertical line passes through more than one point, this graph fails the vertical line test. So, the graph does not represent a function.

3. A function can be described using a ***function rule***, which is an equation that represents an output value, or element of the range, in terms of an input value, or element of the domain.

A function rule can be written in ***function notation***. Here is an example of a function rule and its notation.

$y = 3x + 5$	y is the output and x is the input.
$f(x) = 3x + 5$	Read as "f of x" or "function of f of x"
$f(2) = 3(2) + 5$	"f of 2" is the output when 2 is the input.

Do not confuse parentheses used in function notation with multiplication.

Note that all functions have a corresponding graph. The points that lie on the graph of a function are formed using input values, or elements of the domain as the x-coordinates, and output values, or elements of the range as the y-coordinates.

Example:

Given $f(x) = 2x - 1$, find $f(7)$.

Solution:

$f(7) = 2(7) - 1 = 14 - 1 = 13.$

Example:

If $g(6) = 3 - 5(6)$, find $g(x)$.

Solution:

$g(x) = 3 - 5x$

Example:

If $f(-2) = -4(-2)$, find $f(b)$.

Solution:

$f(b) = -4b$

Example:

Graph $f(x) = 2x - 1$.

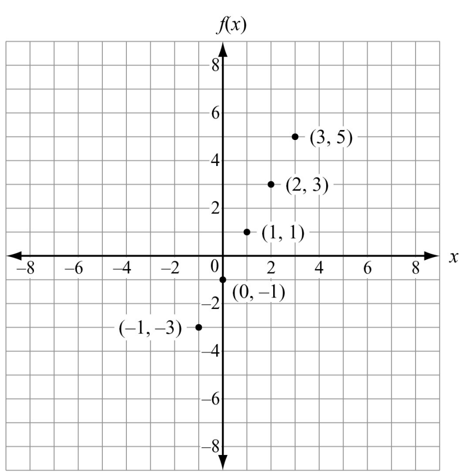
Solution:

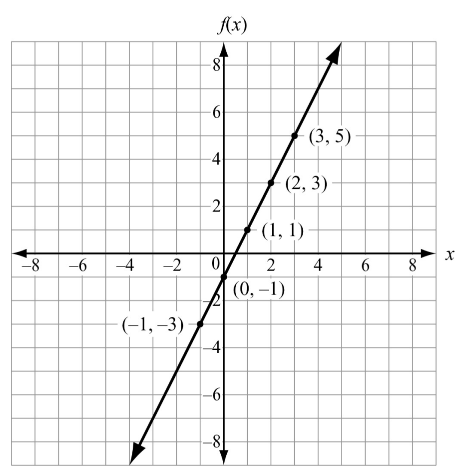
In the function rule $f(x) = 2x - 1$, $f(x)$ is the same as y.

Then we can make a table of x (input) and y (output) values.

x	$f(x)$
–1	–3
0	–1
1	1
2	3
3	5

The values in the rows of the table form ordered pairs. We plot those ordered pairs. If the domain is not specified, we connect the points. If the numbers in the domain are not specified, we assume that it is any real number.





4. A ***sequence*** is an ordered list of numbers. Each number in the sequence is called a term. The terms are consecutive or identified as the first term, second term, third term, and so on. The pattern in the sequence is revealed in the relationship between each term and its term number, or in a term's relationship to the previous term in the sequence.

Example:

Consider the sequence: 3, 6, 9, 12, 15, . . . The first term is 3, the second term is 6, the third term is 9, and so on. The ellipses at the right of the sequence indicate the pattern continues without end. Can this pattern be considered a function?

Solution:

There are different ways of seeing a pattern in the sequence above. One way is to say each number in the sequence is 3 times the number of its term. For example, the fourth term would be 3 times 4, or 12. Looking at the pattern in this way, all you would need to know is the number of the term, and you could predict the value of the term. The value of each term would be a function of its term number. We could use this relationship to write an algebraic rule for the sequence, $y = 3x$, where x is the number of the term and y is the value of the term. This algebraic rule would only assign one number to each input value from the numbers 1, 2, 3, etc. So, we could write a function for the sequence. We can call the function S and write its rule as $S(n) = 3n$, where n is the term number. The domain for the function S would be counting numbers. The range would be the value of the terms in the sequence. When an equation with the term number as a variable is used to describe a sequence, we refer to it as the ***explicit formula*** for the sequence, or the ***closed form***.

Another way to describe the sequence in the example above is to say each term is three more than the term before it. Instead of using the number of the term, you would need to know a previous term to find a subsequent term's value.

Example:

Consider the sequence: 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ *One way* to look at this pattern is to say each successive term is half the term before it, and the first term is 16. With this approach you could easily determine the terms for a limited or ***finite*** sequence.

Another, less intuitive way would be to notice that each term is 32 times a power of $\frac{1}{2}$. If n represents the number of the term, each term is 32 times $\frac{1}{2}$ raised to the nth power, or $32 \times \frac{1}{2}^{n}$. This approach lends itself to finding an explicit formula for any missing term if you know its term number. That is, the value of each term would depend on the term number.

Also, the patterns in sequences can be shown by using tables. Here is how the sequence from the above example would look in table format.

Term Number (n)	1	2	3	4	5	6	7	8
Value (a_n)	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Notice the numbers in the top row of the table are consecutive counting numbers starting with one and increasing to the right. The sequence has eight terms, with 16 being the value of the first term and $\frac{1}{8}$ being the value of the eighth term. A sequence with a specific number of terms is finite. If a sequence continues indefinitely, it is called an ***infinite sequence***.

5. If the nth term of a sequence and the common difference between consecutive terms is known, you can find the $(n + 1)$th term using the ***recursive formula*** $a_n = a_{n-1} + d$, where a_n is the nth term, n is the number of a term, $n - 1$ is the number of the previous term, and d is the common difference.

Take the sequence 3, 6, 9, 12, 15, . . . as an example. We can find the sixth term of the sequence using the fourth and fifth terms.

The common difference d is $15 - 12 = 3$. So, the sixth term is given by $a_6 = a_5 + 3$.
$a_6 = 15 + 3 = 18$

Important Tips

- Do extensive graphing by hand.
- Be extremely careful in the use of language. Always use the name of the function. For example, use f to refer to the function as a whole and use $f(x)$ to refer to the output when the input is x. For example, when language is used correctly, a graph of the function f in the xy-plane is the graph of the equation $y = f(x)$, since only those points of the form (x, y) or $(x, f(x))$ where the y-coordinates are equal to $f(x)$ are graphed.
- Not all sequences can be represented as functions. Be sure to check all the terms you are provided with before reaching the conclusion that there is a pattern.

REVIEW EXAMPLES

1) A manufacturer keeps track of her monthly costs by using a "cost function" that assigns a total cost for a given number of manufactured items, x. The function is $C(x) = 5{,}000 + 1.3x$.

   a. Can any value be in the domain for this function?

   b. What is the cost of 2,000 manufactured items?

   c. If costs must be kept below \$10,000 this month, what is the greatest number of items she can manufacture?

Solution:

a. Since x represents a number of manufactured items, it cannot be negative, nor can a fraction of an item be manufactured. Therefore, the domain can only include values that are whole numbers.

b. Substitute 2,000 for x: $C(2{,}000) = 5{,}000 + 1.3(2{,}000) = \$7{,}600$

c. Form an inequality:

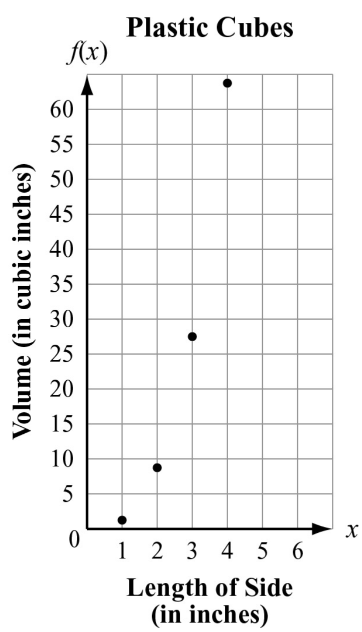
$$C(x) < 10{,}000$$

$$5{,}000 + 1.3x < 10{,}000$$

$$1.3x < 5{,}000$$

$$x < 3{,}846.2\text{, or } 3{,}846 \text{ items}$$

2) A company makes plastic cubes with sides that have lengths of 1 inch, 2 inches, 3 inches, or 4 inches. The function $f(x) = x^3$ represents the relationship between x, the side length of the cube, and $f(x)$, the volume of the cube. The graph shown below represents the function.



a. What is the domain of the function?

b. What is the range of the function?

c. If the company decided to make cubes with a side length of m inches, what would be the volume of those cubes?

Solution:

a. $\{1, 2, 3, 4\}$

b. $\{1, 8, 27, 64\}$

c. The volume would be m^3 cubic inches.

Note: The points start at (1, 0), because 1 is the smallest length of a side of a plastic cube. They are not connected because that would imply that the lengths of the cubes could include numbers that lie between the whole numbers that are given in the context of the problem. This is called a ***discrete*** function.

3) Consider this sequence: 5, 7, 11, 19, 35, 67, . . .

a. Is this a finite sequence or an infinite sequence?

b. What is a_1? What is a_3?

c. What is the domain of the sequence? What is the range?

Solution:

a. The ellipsis at the end of the sequence indicates that it is an infinite sequence.

b. a_1 is 5 and a_3 is 11.

c. The domain is $\{1, 2, 3, 4, 5, 6 \ldots\}$, and the range is $\{5, 7, 11, 19, 35, 67, \ldots\}$.

Note that this sequence has a pattern that can be expressed using the recursive definition $a_1 = 5,\ a_n = 2a_{n-1} - 3$.

3) The function $f(n) = -(1-4n)$ represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

Solution:

n	1	2	3	4	5
***f*(*n*)**	3	7	11	15	19

Since the function is a sequence, the domain would be n, the number of each term in the sequence. The set of numbers in the domain can be written as $\{1, 2, 3, 4, 5, \ldots\}$. Notice that the domain is an infinite set of numbers, even though the table only displays the first five elements.

The range is $f(n)$ or (a_n), the output numbers that result from applying the rule $-(1-4n)$. The set of numbers in the range, which is the sequence itself, can be written as {3, 7, 11, 15, 19,…}. This is also an infinite set of numbers, even though the table only displays the first five elements.

EOCT Practice Items

1) The first term in this sequence is –1.

n	1	2	3	4	5	. . .
a_n	–1	1	3	5	7	. . .

Which function represents the sequence?

A. $n + 1$
B. $n + 2$
C. $2n - 1$
D. $2n - 3$

[Key: D]

2) Which function is modeled in this table?

x	$f(x)$
1	8
2	11
3	14
4	17

A. $f(x) = x + 7$

B. $f(x) = x + 9$

C. $f(x) = 2x + 5$

D. $f(x) = 3x + 5$

[Key: D]

3) Which explicit formula describes the pattern in this table?

d	C
2	6.28
3	9.42
5	15.70
10	31.40

A. $d = 3.14 \times C$

B. $3.14 \times C = d$

C. $31.4 \times 10 = C$

D. $C = 3.14 \times d$

[Key: D]

4) If $f(12) = 4(12) – 20$, which function gives $f(x)$?

A. $f(x) = 4x$

B. $f(x) = 12x$

C. $f(x) = 4x – 20$

D. $f(x) = 12x – 20$

[Key: C]

INTERPRET FUNCTIONS THAT ARISE IN APPLICATIONS IN TERMS OF THE CONTEXT

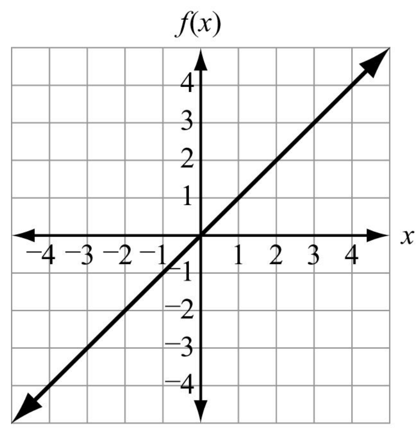
KEY IDEAS

1. Through examining the graph of a function, many of its features are discovered. Features include domain and range, *x*- and *y*- intercepts, intervals where the function values are increasing or decreasing, positive or negative, minimums or maximums, and rates of change.

Example:

Consider the graph of $f(x)$ below. It appears to be a line, unbroken and slanted upward.

Linear Function
$\boldsymbol{f(x) = x}$

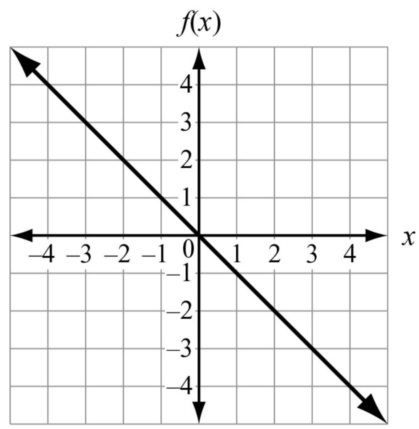


Some of its key features are:

- Domain: All real numbers
- Range: All real numbers
- *x*-intercept: The line appears to intersect the *x*-axis at 0.
- *y*-intercept: The line appears to intersect the *y*-axis at 0.
- Increasing: As x increases, $f(x) > 0$
- Decreasing: Never
- Positive: $f(x) > 0$ when $x > 0$
- Negative: $f(x) < 0$ when $x < 0$
- Minimum or Maximum: None
- Rate of change: 1

Example:

Consider the graph of $f(x) = -x$. It appears to be an unbroken line and slanted downward.

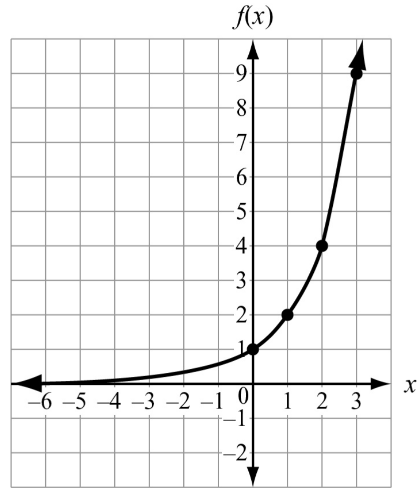


- Domain: All real numbers because there is a point on the graph for every possible x-value.
- Range: All real numbers because there is a point on the graph that corresponds to every possible y-value.
- x-intercept: It appears to intersect the x-axis at 0.
- y-intercept: It appears to intersect the x-axis at 0.
- Increasing: The function does not increase.
- Decreasing: The graph falls from left to right.
- Positive: $f(x)$ is positive for $x < 0$.
- Negative: $f(x)$ is negative for $x > 0$.
- Minimums or Maximums: None
- Rate of change: –1

Example:

Consider the graph of $f(x) = 2^x$.

Exponential Function
$\boldsymbol{f(x) = 2^x}$



- Domain: All real numbers because there is a point on the graph for every possible x-value.
- Range: $y > 0$
- x-intercept: None
- y-intercept: It appears to intersect the y-axis at 1.
- Increasing: Always
- Decreasing: Never
- Positive: $f(x)$ is positive for all x values
- Negative: $f(x)$ is never negative.
- Minimum or Maximum: None
- Rate of change: It appears to vary as the graph has curvature and is not straight.

2. Other features of functions can be discovered through examining their tables of values. The intercepts may appear in a table of values. From the differences of $f(x)$ values over various intervals we can tell if a function grows at a constant rate of change.

Example:

Let $h(x)$ be the number of person-hours it takes to assemble x engines in a factory. The company's accountant determines that the time it takes depends on start-up time and the number of engines to be completed. It takes 6.5 hours to set up the machinery to make the engines and about 5.25 hours to completely assemble one. The relationship is modeled with the function $h(x) = 6.5 + 5.25x$. Next, he makes a table of values to check his function against his production records. He starts with 0 engines because of the startup time. The realistic domain for the accountant's function would be whole numbers, because you cannot manufacture a negative number of engines.

x (engines)	$h(x)$ hours
0	6.5
1	11.75
2	17
3	22.25
4	27.5
5	32.75
10	59
100	531.5

From the table we can see the y-intercept. The y-intercept is the y-value when $x = 0$. The very first row of the table indicates the y-intercept is 6.5. Since we do not see a number 0 in the h(x) column, we cannot tell from the table if there is an x-intercept. The x-intercept is the value when $h(x) = 0$.

$$\begin{aligned} h(x) &= 6.5 + 5.25x \\ 0 &= 6.5 + 5.25x \\ -6.5 &= 5.25x \\ -1.24 &= x \end{aligned}$$

The x-value when $y = 0$ is negative, which is not possible in the context of this example.

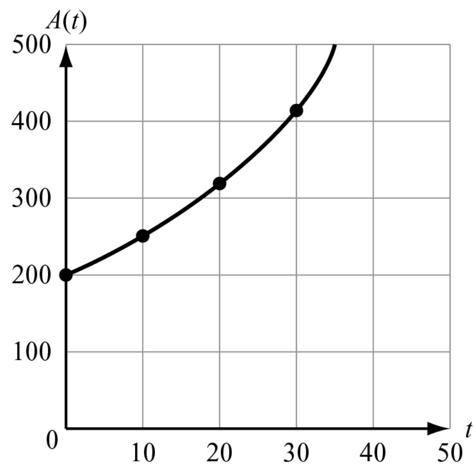
The accountant's table also gives us an idea of the rate of change of the function. We should notice that when x values are increasing by 1, the $h(x)$ values are growing by increments of 5.25. There appears to be a constant rate of change when the input values increase by the same amount. The increase from both 3 engines to 4 engines and 4 engines to 5 engines is 5.25 hours. We can calculate the average rate of change by comparing the values in the first and last rows of the table. The increase in number of engines made is 100 – 0, or 100. The increase in hours is 531.5 – 6.5, or 525. The average rate of change is $\frac{525}{100} = 5.25$. The units for this average rate of change would be hours/engine, which happens to be the exact amount of time it takes to make an engine.

Important Tips

- Begin exploration of a new function by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.
- Do not trust apparent intersections on graphs. Always check the coordinates in the equation.

REVIEW EXAMPLES

1) The amount accumulated in a bank account over a time period t and based on an initial deposit of \$200 is found using the formula $A(t) = 200(1.025)^t$, $t \geq 0$. Time, t, is represented on the horizontal axis. The accumulated amount, $A(t)$, is represented on the vertical axis.



a. What are the intercepts of the function $A(t)$?

b. What is the domain of the function $A(t)$?

c. Why are all the t values non-negative?

d. What is the range of $A(t)$?

e. Does $A(t)$ have a maximum or minimum value?

Solution:

a. $A(t)$ has no t-intercept. $A(t)$ crosses the vertical axis at 200.

b. The domain for $A(t)$ is $t \geq 0$.

c. The t values are all non-negative because they represent time, and time cannot be negative.

d. $A(t) \geq 200$.

e. The function $A(t)$ has no maximum value. Its minimum value is 200.

2) A company uses the function $V(x) = 28{,}000 - 1{,}750x$ to represent the depreciation of a truck, where V is the value of the truck and x is the number of years after its purchase. Use the table of values shown below.

x, years	**$V(x)$, (value in \$)**
0	28,000
1	26,250
2	24,500
3	22,750
4	21,000
5	19,250

a. What is the y-intercept of the graph of the function?

b. Does the graph of the function have an x-intercept?

c. Does the function increase or decrease?

Solution:

a. From the table, when $x = 0$, $V(x) = 28{,}000$. So, the y-intercept is 28,000.

b. Yes, it does have an x-intercept, although it is not shown in the table. The x-intercept is the value of x when $V(x) = 0$.

$$\begin{aligned} 0 &= 28{,}000 - 1{,}750x \\ -28{,}000 &= -1{,}750x \\ 16 &= x \end{aligned}$$

The x-intercept is 16.

c. As $x > 0$, $V(x)$ decreases. So, the function decreases.

EOCT Practice Items

1) A farmer owns a horse that can continuously run an average of 8 miles an hour for up to 6 hours. Let *y* be the distance the horse can travel for a given *x* amount of time in hours. The horse's progress can be modeled by a function.

Which of the following describes the domain of the function?

A. $0 \leq x \leq 6$

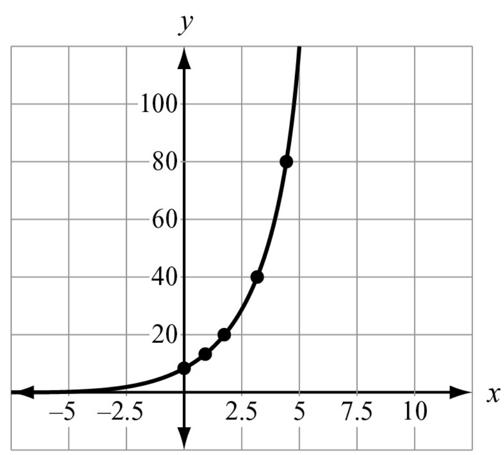
B. $0 \leq y \leq 6$

C. $0 \leq x \leq 48$

D. $0 \leq y \leq 48$

[Key: A]

2) A population of squirrels doubles every year. Initially there were 5 squirrels. A biologist studying the squirrels created a function to model their population growth, $P(t) = 5(2^t)$ where *t* is time. The graph of the function is shown. What is the range of the function?



A. any real number

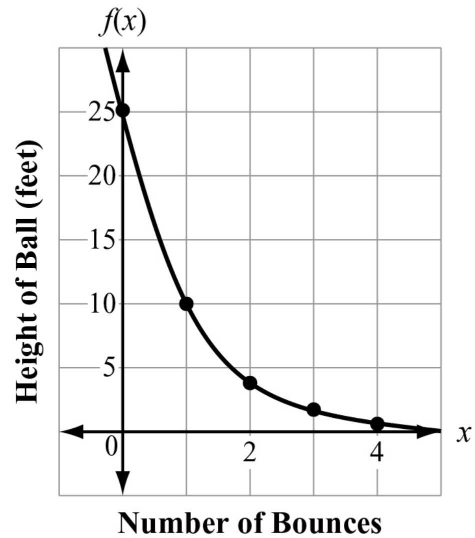
B. any whole number greater than 0

C. any whole number greater than 5

D. any whole number greater than or equal to 5

[Key: D]

3) The function graphed on this coordinate grid shows *y*, the height of a dropped ball in feet after its *x*th bounce.



On which bounce was the height of the ball 10 feet?

A. bounce 1

B. bounce 2

C. bounce 3

D. bounce 4

[Key: A]

ANALYZE FUNCTIONS USING DIFFERENT REPRESENTATIONS

KEY IDEAS

1. When working with functions, it is essential to be able to interpret the specific quantitative relationship regardless of the manner of its presentation. Understanding different representations of functions such as tables, graphs, and equations makes interpreting relationships between quantities easier. Beginning with lines, we will learn how each representation aids our understanding of a function. Almost all lines are functions, except vertical lines, because they assign multiple elements of their range to just one element in their domain. All linear functions can be written in the form $f(x) = ax + b$, where a and b are real numbers and x is a variable to which the function f assigns a corresponding value, $f(x)$.

Example:

Consider the linear functions $f(x) = x + 5$, $g(x) = 2x – 5$, and $h(x) = –2x$.

First we will make a table of values for each equation. To begin, we need to decide on the domains. In theory, $f(x)$, $g(x)$, and $h(x)$ can accept any number as input. So, the three of them have all real numbers as their domains. But, for a table we can only include a few elements of their domains. We should choose a sample that includes negative numbers, 0, and positive numbers. Place the elements of the domain in the left column, usually in ascending order. Then apply the function's assignment rule to determine the corresponding element in the range. Place it in the right column.

x	$f(x) = x + 5$
–3	2
–2	3
–1	4
0	5
1	6
2	7
3	8
4	9

x	$g(x) = 2x – 5$
–3	–11
–2	–9
–1	–7
0	–5
1	–3
2	–1
3	1
4	3

x	$h(x) = –2x$
–3	6
–2	4
–1	2
0	0
1	–2
2	–4
3	–6
4	–8

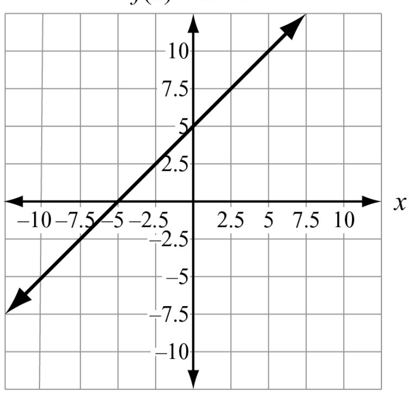
We can note several features about the functions just from their table of values.

- $f(x)$ has a y-intercept of 5. When x is 0, $f(x) = 5$. It is represented by (0, 5) on its graph.
- $g(x)$ has a y-intercept of –5. When x is 0, $g(x) = –5$. It is represented by (0, –5) on its graph.

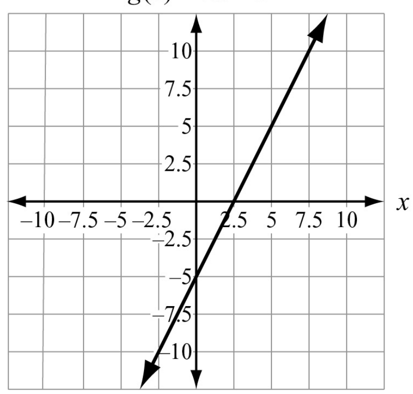
- $h(x)$ has a y-intercept of 0. When x is 0, $h(x) = 0$. It is represented by (0, 0) on its graph.
- $h(x)$ has a x-intercept of 0. When $h(x) = 0$, $x = 0$. It is represented by (0, 0) on its graph.
- $f(x)$ has an average rate of change of 1. $\frac{9-2}{4-(-3)} = 1$
- $g(x)$ has an average rate of change of 2. $\frac{3-(-11)}{4-(-3)} = 2$
- $h(x)$ has an average rate of change of –2. $\frac{(-8)-6}{4-(-3)} = -2$

Now we will take a look at the graphs of $f(x)$, $g(x)$, and $h(x)$.

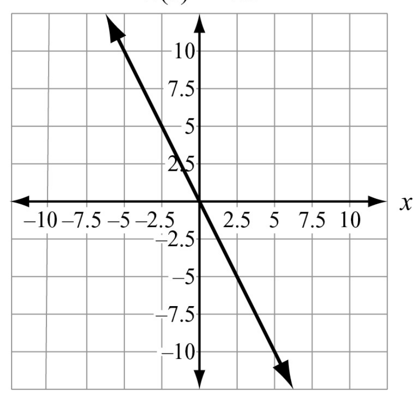
$f(x) = x + 5$



$g(x) = 2x – 5$



$h(x) = –2x$



Their graphs confirm what we already learned about their intercepts and their constant rates of change. The graphs suggest other information:

- $f(x)$ appears to have positive values for $x > –5$ and negative values for $x < –5$.
- $f(x)$ appears to be always increasing with no maximum or minimum values.
- $g(x)$ appears to have positive values for $x > 2.5$ and negative values for $x < 2.5$.
- $g(x)$ appears to be always increasing with no maximum or minimum values.
- $h(x)$ appears to have positive values for $x < 0$ and negative values for $x > 0$.
- $h(x)$ appears to be always decreasing with no maximum or minimum values.

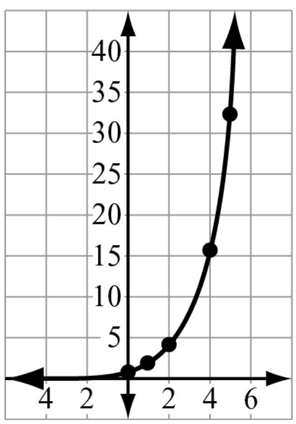
To confirm these observations we can work with the equations for the functions. We suspect $f(x)$ is positive for $x > –5$. Since $f(x)$ is positive whenever $f(x) > 0$, write and solve the inequality $x + 5 > 0$ and solve for x. We get $f(x) > 0$ when $x > –5$. We can confirm all our observations about $f(x)$ from working with the equation. Likewise, the observations about $g(x)$ and $h(x)$ can be confirmed using their equations.

2. The three ways of representing a function also apply to exponential functions. Exponential functions are built using powers. A power is the combination of a base with an exponent. The third power of 5 is written as 5^3. The exponent is the superscripted number written to the right of the base. A power is a shortcut way of writing a product of identical factors. Instead of writing $5 \times 5 \times 5$, we write 5^3. We can also think of the exponent as a count of how the number of times a base is a repeated factor. The base of a power need not be a number. Expressions such as x^5 or $(x + 1)^7$ are also powers. The reason some functions are termed exponential is they have a variable as an exponent. Exponential functions are in a family of functions with similar equations. They are of the form $f(x) = ab^x$ where $a \neq 0$, with $b > 0$, and $b \neq 1$. In an exponential function, the base b is a constant.

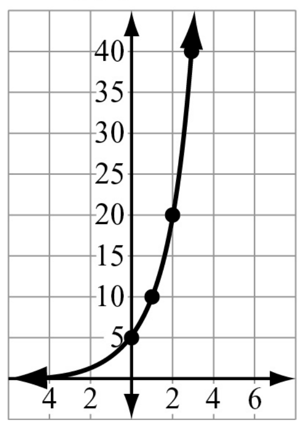
Example:

Consider $f(x) = 2^x$, $g(x) = 5 \cdot 2^x$, and $h(x) = -2^x$. For all three functions, $f(x)$, $g(x)$, and $h(x)$, the base is 2. So, it is the coefficient that causes the graphs to look different.

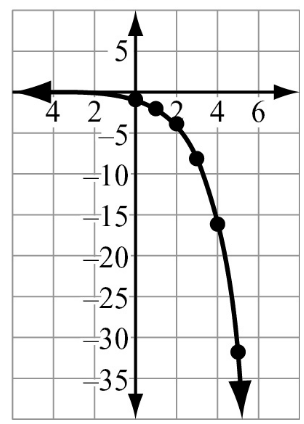
$f(x) = 2^x$



$g(x) = 5 \bullet 2^x$



$h(x) = -2^x$



From the graphs, the following appears:

- $f(x)$ appears to have a y-intercept at 1.
- $g(x)$ appears to have a y-intercept at 5
- $h(x)$ appears to have a y-intercept at –1.
- For $f(x)$, as x increases, $f(x)$ increases and as x decreases, f(x) approaches 0.
- For $g(x)$, as x increases, $g(x)$ increases and as x decreases, $g(x)$ approaches 0.
- For $h(x)$, as x increases, $h(x)$ decreases and as x decreases, $h(x)$ approaches 0.
- None of the functions appear to have a constant rate of change.

Having looked at their graphs, we will now look at their tables.

x	$f(x) = 2^x$
–3	$\frac{1}{8}$
–2	$\frac{1}{4}$
–1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16

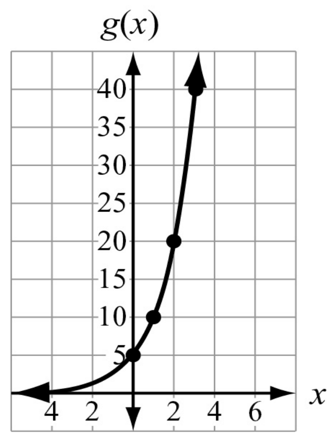
x	$g(x) = 5 \cdot 2^x$
–3	$\frac{5}{8}$
–2	$\frac{5}{4}$
–1	$\frac{5}{2}$
0	5
1	10
2	20
3	40
4	80

x	$h(x) = -2^x$
–3	$-\frac{1}{8}$
–2	$-\frac{1}{4}$
–1	$-\frac{1}{2}$
0	–1
1	–2
2	–4
3	–8
4	–16

The tables confirm all three functions have y-intercepts: $f(0) = 1$, $g(0) = 5$, and $h(0) = -1$. Although the tables do not show a constant rate of change for any of the functions, a constant rate of change can be determined on a specific interval. We will not bother to calculate an average rate of change.

3. Comparing functions helps us gain a better understanding of them. Let's take a look at a linear function and the graph of an exponential function.

Consider $f(x) = 2x$ and the graph of $g(x)$ below. The function $f(x) = 2x$ represents a linear relationship. This graph shows an exponential relationship. We know the linear function has a graph that is a straight line and has a constant rate of change. The graph of the exponential function is curved and has a varying rate of change. Both curves will have y-intercepts. The exponential curve appears to have a y-intercept below 5. The y-intercept of the line is $f(0) = 2 \cdot 0 = 0$. Since $f(0) = 0$, the line must pass through the point (0, 0), so 0 must also be the x-intercept of the line also. The graph of the exponential function does not appear to have an x-intercept, though the curve appears to come very close to the x-axis.



For $f(x) = 2x$, the domain and range are all real numbers. For the exponential function, the domain is defined for all real numbers and the range is defined for positive values.

So, while the two graphs share some features, they also have significant differences; the most important of which is that one is a straight line and has a constant rate of change, while the exponential function is curved with a rate of change that is increasing.

Important Tips

- Remember the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function.
- Be familiar with important features of a function such as intercepts, domain, range, minimum and maximums, and periods of increasing and decreasing values.

REVIEW EXAMPLES

1) What are the key features of the function $p(x) = \frac{1}{2}x - 3$?

Solution:

First, notice that the function is linear. The domain for the function is the possible numbers we can substitute for x. Since the function is linear, the domain is all real numbers. The graphic representation will give us a better idea of its range.

We can determine the y-intercept by finding $p(0)$:

$$p(0) = \frac{1}{2}(0) - 3 = -3$$

So, the graph of $p(x)$ will intersect the y-axis at (0, –3). To find the x-intercept, we have to solve the equation $p(x) = 0$.

$$\begin{aligned} \frac{1}{2}x - 3 &= 0 \\ \frac{1}{2}x &= 3 \\ x &= 6 \end{aligned}$$

So, the x-intercept is 6. The line intersects the x-axis at (6, 0).

Now we will make a table of values to investigate the rate of change of $f(x)$. We can use $x = -3$ and $x = 4$.

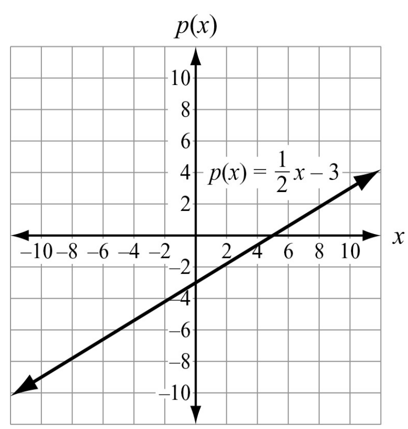
x	$p(x)=\frac{1}{2}x-3$
–3	$-\frac{9}{2}$
–2	– 4
–1	$-\frac{7}{2}$
0	–3
1	$-\frac{5}{2}$
2	–2
3	$-\frac{3}{2}$
4	–1

Notice the row that contains the values 0 and –3. These numbers correspond to the point where the line intersects the y-axis, confirming that the y-intercept is –3. Since 0 does not appear in the right column, the coordinates of the x-intercept point are not in the table of values. We notice that the values in the right column keep increasing by $\frac{1}{2}$. We can calculate the average rate of change.

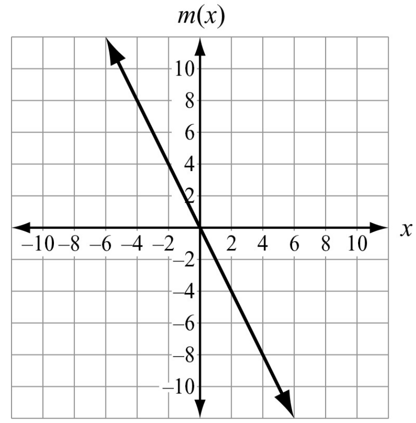
Average rate of change: $\dfrac{-1-\frac{-9}{2}}{4-(-3)}=\dfrac{1}{2}$

It turns out the average rate of change is the same as the incremental differences in the outputs. This confirms the function $p(x)$ has a constant rate of change. Notice that $\frac{1}{2}$ is the coefficient of x in the function rule.

Now we will examine the graph. The graph shows a line that appears to be always increasing. Since the line has no minimum or maximum value, its range would be all real numbers. The function appears to have positive values for $x > 6$ and negative values for $x < 6$.



2) Compare $p(x) = \frac{1}{2}x - 3$ from the previous example with the function $m(x)$ in the graph below.

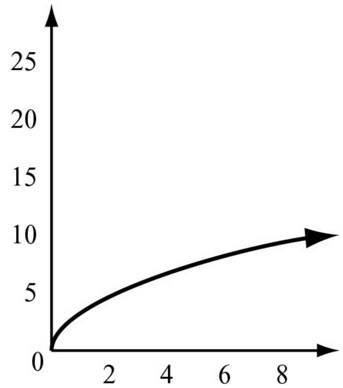


The graph of $m(x)$ intersects both the x- and y-axes at 0. It appears to have a domain of all real numbers and a range of all real numbers. So, $m(x)$ and $p(x)$ have the same domain and range. The graph appears to have a constant rate of change and is decreasing. It has positive values when $x < 0$ and negative values when $x > 0$.

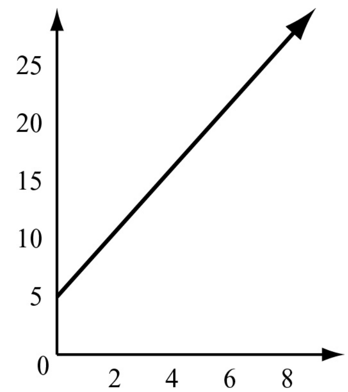
<u>EOCT Practice Items</u>

1) To rent a canoe, the cost is \$3 for the oars and life preserver, plus \$5 an hour for the canoe. Which graph models the cost of renting a canoe?

A



B



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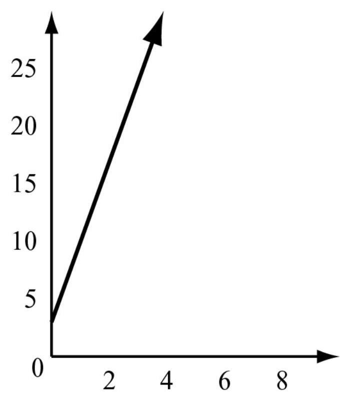

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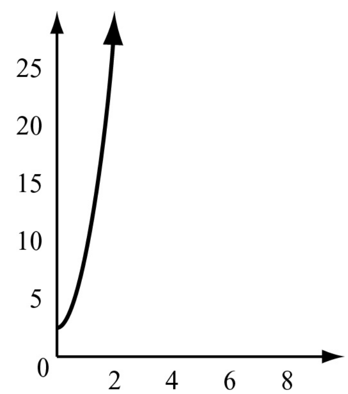
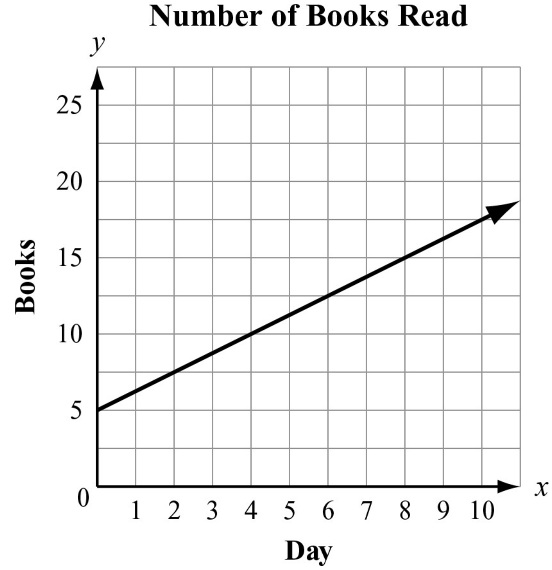


[Key: C]

2) Juan and Patti decided to see who could read the most books in a month. They began to keep track after Patti had already read 5 books that month. This graph shows the number of books Patti read for the next 10 days.



If Juan has read no books before the fourth day of the month and he reads at the same rate as Patti, how many books will he have read by day 12?

A. 5

B. 10

C. 15

D. 20

[Key: B]

BUILD A FUNCTION THAT MODELS A RELATIONSHIP BETWEEN TWO QUANTITIES

KEY IDEAS

1. Modeling a quantitative relationship can be a challenge. But there are some techniques we can use to make modeling easier. Mostly, we decide which kind of model to use based on the rate of change of the function's values.

Example:

Joe started with \$13. He has been saving \$2 each week to purchase a baseball glove. The amount of money Joe has depends on how many weeks he has been saving. So, the number of weeks and the amount Joe has saved are related. We can begin with the function $S(x)$, where S is the amount he has saved and x is the number of weeks. Since we know that he started with \$13 and that he saves \$2 each week we can use a linear model, one where the change is constant.

A linear model for a function is $f(x) = ax + b$, where a and b are any real numbers and x is the independent variable.

So, the model is $S(x) = 2x + 13$ which will generate the amount Joe has saved after x weeks.

Example:

Pete withdraws half his savings every week. If he started with \$400, can we write a rule for how much Pete has left each week? We know the amount Pete has left depends on the week. Once again we can start with the amount Pete has, $A(x)$. The amount depends on the week number, x. However, this time the rate of change is not constant. Therefore, the previous method for finding a function will not work. We could set up the model as $A(x) = 400 \times \frac{1}{2} \times \ldots \times \frac{1}{2}$ and use $\frac{1}{2}$ as the number of weeks, x.

Or, we can use a power of $\frac{1}{2}$:

$$A(x) = 400 \times \left(\frac{1}{2}\right)^x$$

Note that the function assumes Pete had \$400 at week 0 and withdraws half during week 1. The exponential function will generate the amount Pete has after x weeks.

2. Sometimes the data for a function is presented as a sequence.

Example:

Suppose we know the total number of cookies eaten by Rachel on a day-to-day basis over the course of a week. We might get a sequence like this: 3, 5, 7, 9, 11, 13, 15. There are two ways we could model this sequence. The first would be the explicit way. We would arrange the sequence in a table. Note that the symbol Δ in the third row means change or difference.

n	1	2	3	4	5	6	7
a_n	3	5	7	9	11	13	15
Δa_n	---------	5 – 3 = 2	7 – 5 = 2	9 – 7 = 2	11 – 9 = 2	13 – 11 = 2	15 – 13 = 2

Since the difference between successive terms of the sequence is constant, namely 2, we can again use a linear model. But this time we do not know the y-intercept because there is no zero term ($n = 0$). However, if we work backward, a_0—the term before the first—would be 1, so the starting number would be 1. That leaves us with an explicit formula: $f(n) = 2n + 1$, for $n > 0$ (n is an integer). A sequence that can be modeled with a linear function is called an ***arithmetic sequence***.

Another way to look at the sequence is recursively. We need to express term n (a_n) in terms of a previous term. Since the constant difference is 2, we know $a_n = a_{n-1} + 2$ for $n > 1$, with $a_1 = 3$.

3. Some sequences can be modeled exponentially. For a sequence to fit an exponential model, the ratio of successive terms is constant.

Example:

Consider the number of sit-ups Clara does each week as listed in the sequence 3, 6, 12, 24, 48, 96, 192. Clara is doing twice as many sit-ups each successive week. It might be easier to put the sequence in a table to analyze it.

n	1	2	3	4	5	6	7
a_n	3	6	12	24	48	96	192
a_n/a_{n-1}	-----	6/3 = 2	12/6 = 2	24/12 = 2	48/24 = 2	96/48 = 2	192/96 = 2

It appears as if each term is twice the term before it. But the difference between the terms is not constant. This type of sequence shows exponential growth. The function type is $f(x) = a(b^x)$. In this type of function, b is the coefficient and b^x is the growth power. For the sequence above, the growth power is 2^x because the terms keep doubling. To find b you need to know the first term. The first term is 3. The second term is the first term of the sequence multiplied by the common ratio once. The third term is the first term multiplied by the common ratio twice. Since that pattern continues, our exponential function is $f(x) = 3(2^{x-1})$. The function $f(x) = 3(2^{x-1})$ would be the explicit or ***closed form*** for the sequence. A sequence that can be modeled by an exponential function is a ***geometric sequence***.

The sequence could also have a recursive rule. Since the next term is twice the previous term, the recursive rule would be $a_n = 2 \cdot a_{n-1}$, with a first term, a_1, of 3.

4. Exponential functions have lots of practical uses. They are used in many real-life situations.

Example:

A scientist collects data on a colony of microbes. She notes these numbers:

Day	1	2	3	4	5	6
a_n	800	400	200	100	50	25
a_n/a_{n-1}	-----	400/800 = 0.5	200/400 = 0.5	100/200 = 0.5	50/100 = 0.5	25/50 = 0.5

Since the ratio between successive terms is a constant 0.5, she believes the growth power is (0.5^{x-1}), where x is the number of the day. She uses 800 for the coefficient, since that was the population on the first day. The function she used to model the population size was $f(x) = 800(0.5^{x-1})$.

Important Tips

- Examine function values to draw conclusions about the rate of change.
- Keep in mind the general forms of a linear function and exponential function.

REVIEW EXAMPLES

1) The terms of a sequence increase by a constant amount. If the first term is 7 and the fourth term is 16:

   a. List the first six terms of the sequence.

   b. What is the explicit formula for the sequence?

   c. What is the recursive rule for the sequence?

Solution:

a. The sequence would be: 7, 10, 13, 16, 19, 22, . . . $\frac{16-7}{4-3} = \frac{9}{3} = 3.$ If the difference between the first and fourth terms is 9, the constant difference is 3. So, the sequence is arithmetic.

b. Since the constant difference is 3, $a = 3$. Because the first term is 7, $b = 7 - 3 = 4$. So, the explicit formula is: $f(n) = 3(n) + 4$, for $n > 0$.

c. Since the difference between successive terms is 3, $a_n = a_{n-1} + 3$ with $a_1 = 7$.

2) The function $f(n) = -(1 - 4n)$ represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

Solution:

n	1	2	3	4	5
$f(n)$	3	7	11	15	19

Since the function is a sequence, the domain would be n, the number of each term in the sequence. The set of numbers in the domain can be written as $\{1, 2, 3, 4, 5, \ldots\}$. Notice that the domain is an infinite set of numbers, even though the table lists only the first five.

The range is $f(n)$ or (a_n), the output numbers that result from applying the rule $4n - 1$. The set of numbers in the range, which is the sequence itself, can be written as $\{3, 7, 11, 15, 19, \ldots\}$. This is also an infinite set of numbers, even though the table lists only the first five.

EOCT Practice Items

1) Which function represents this sequence?

n	1	2	3	4	5	…
a_n	6	18	54	162	486	…

A. $f(n) = 3^{n-1}$
B. $f(n) = 6^{n-1}$
C. $f(n) = 3(6^{n-1})$
D. $f(n) = 6(3^{n-1})$

[Key: D]

2) The first term in this sequence is 3.

n	1	2	3	4	5	. . .
a_n	3	10	17	24	31	. . .

Which function represents the sequence?

A. $f(n) = n + 3$
B. $f(n) = 7n - 4$
C. $f(n) = 3n + 7$
D. $f(n) = n + 7$

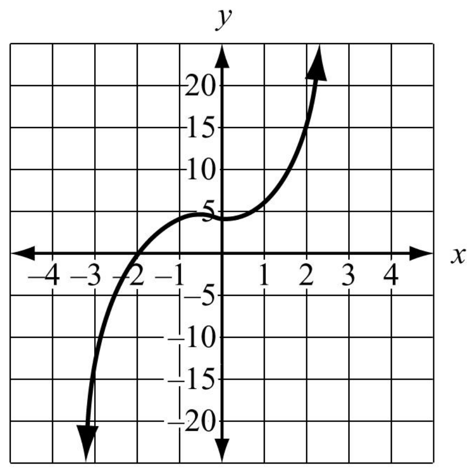
[Key: B]

3) The points (0, 1), (1, 5), (2, 25), (3, 125) are on the graph of a function. Which equation represents that function?

A. $f(x) = 2^x$
B. $f(x) = 3^x$
C. $f(x) = 4^x$
D. $f(x) = 5^x$

[Key: D]

4) The graph of a function is shown on this coordinate plane.



Which statement best describes the behavior of the function within the interval $x = -3$ to $x = 0$?

A. From left to right, the function rises only.
B. From left to right, the function falls and then rises.
C. From left to right, the function rises and then falls.
D. From left to right, the function falls, rises, and then falls.

[Key: C]

BUILD NEW FUNCTIONS FROM EXISTING FUNCTIONS

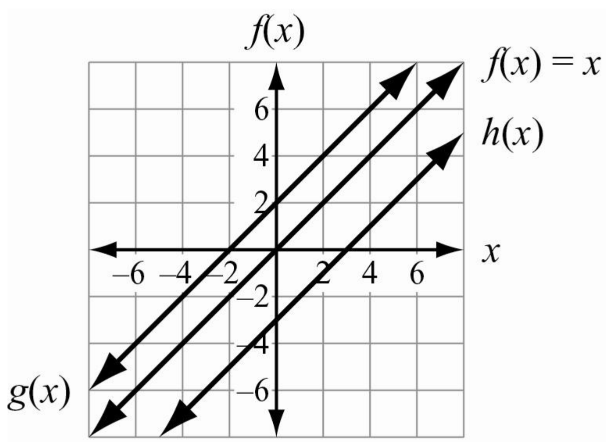
KEY IDEAS

1. Functions can be transformed in many ways. Whenever a function rule is transformed, the transformation affects the graph. One way to transform a function is to increase or decrease the assigned values by a specified amount. Such adjustments have the effect of shifting the function's graph up or down. These shifts are called ***translations***. The original curve is moved from one place to another on the coordinate plane the same as translations in geometry.

Example:

If $f(x) = x$, how will $g(x) = f(x) + 2$ and $h(x) = f(x) - 3$ compare?

Solution:



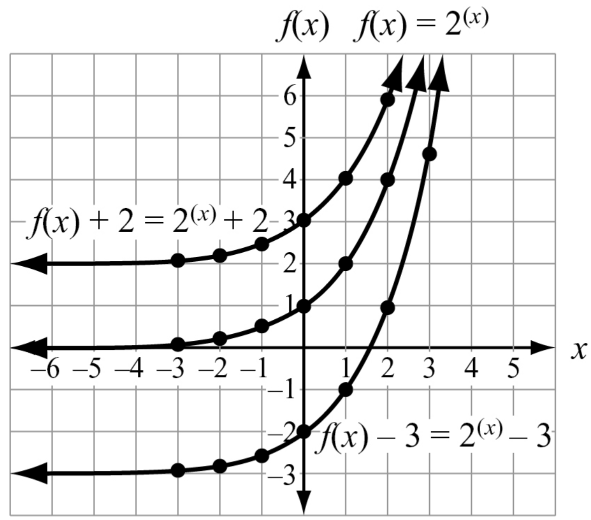
As expected, the lines all have the same shape, rate of change, domain, and range. The adjustments affected the intercepts. When 2 is added to the function's values, the y-intercept is 2 units higher. When 3 is subtracted, the y-intercept drops 3 units. When quantities are added or subtracted from a function's values, it causes a vertical translation of the graph. This effect would be true of all types of functions.

Example:

If $f(x) = 2^x$, how will $f(x) + 2$ and $f(x) - 3$ compare?

Solution:

We need to make the adjustments algebraically first. Using the logic, if $f(x) = 2^{(x)}$, then $f(x) + b = 2^{(x)} + b$. We will compare graphs of $f(x) = 2^{(x)}$, $f(x) + 2 = 2^{(x)} + 2$, and $f(x) - 3 = 2^{(x)} - 3$.



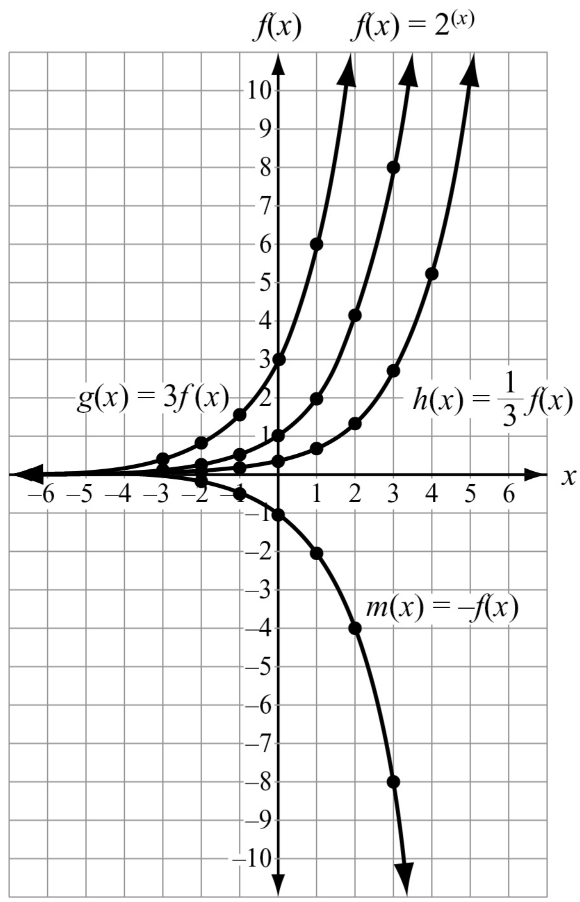
The curves have not changed shape. Their domains and ranges are unchanged. However, the curves are not in the same positions vertically. The function $f(x) + 2 = 2^x + 2$ is a translation of $f(x) = 2^x$ upward by 2 units. The function $f(x) - 3 = 2^x - 3$ is a translation downward by 3 units.

2. Functions can be adjusted by factors as well as sums or subtractions. The factors can affect the functions either before or after they make their assignments. When a function is multiplied by a factor after the value is assigned, it stretches or shrinks the graph of the function. If the factor is greater than 1, it stretches the graph of the function. If the factor is between 0 and 1, it shrinks the graph of the function. If the factor is –1, it reflects the function over the x-axis.

Example:

If $f(x) = 2^{(x)}$, how will $g(x) = 3f(x)$, $h(x) = \frac{1}{3}f(x)$ and $m(x) = -f(x)$ compare?

Solution:



The graphs all have y-intercepts, but the rates of change are affected.

In summary:

- ✓ Adjustments made by adding or subtracting values, either before or after the function, assign values to inputs and cause translations of the graphs.
- ✓ Adjustments made by using factors, either before or after the function, assign values to inputs and affect the rates of change of the functions and their graphs.
- ✓ Multiplying by a factor of –1 reflects a linear function over the x-axis.

4. We call f an even function if $f(x) = f(-x)$ for all values in its domain.

Example:

Suppose f is an even function and the point (2, 7) is on the graph of f. Name one other point that must be on the graph of f.

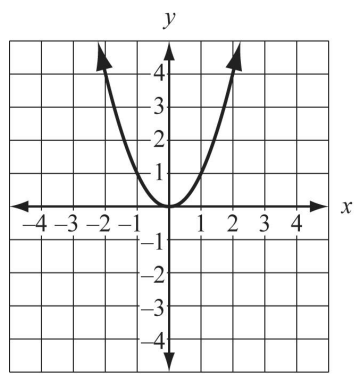
Solution:

Since (2, 7) is on the graph, 2 is in the domain and $f(2) = 7$. By definition of an even function, $f(-2) = f(2) = 7$.

Therefore, (–2, 7) is also on the graph of f.

5. The graph of an ***even function*** has line symmetry with respect to the y-axis.

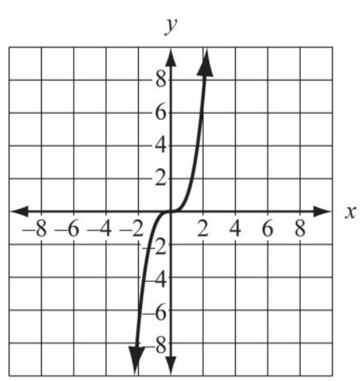
This is the graph of an even function.



Notice that for any number b, the points (x, b) and $(-x, b)$ are at the same height on the grid and are equidistant from the y-axis. That means they represent line symmetry with respect to the y-axis.

6. We call f an ***odd function*** if $f(-x) = -f(x)$ for all values in its domain.

This is the graph of an odd function.



The graph of an odd function has rotational symmetry of 180° about the origin. This is also called symmetry with respect to the origin. Whenever the graph of an odd function contains the point (a, b) it also contains the point $(-a, -b)$.

Example:

Suppose f is an odd function and the point $(-2, 8)$ is on the graph of f. Name one other point that must be on the graph of f.

Solution:

Since $(-2, 8)$ is on the graph, -2 is in the domain and $f(-2) = 8$. By definition of an odd function, $-(-2)$, or 2, is also in the domain and $f(2) = -f(-2) = -8$.

Therefore, $(2, -8)$ is also on the graph of f.

Important Tips

- Remember that a horizontal transformation affects the elements of the domain of a function and a vertical transformation affects the elements of the range of a function.
- Graph transformed functions in the same coordinate plane to see how their graphs compare.

REVIEW EXAMPLES

1) For the function $f(x) = 3^x$:

   a. Find the function that represents a 5 unit translation upward of the function.

   b. Find the function that represents a 3 unit translation to the left of the function.

   c. Is the function even, odd, or neither even nor odd?

Solution:

a. $f(x) = 3^x + 5$
b. $f(x + 3) = 3^{x+3}$
c. $f(x)$ is neither even nor odd. It has no particular symmetry.

2) Given the function $f(x) = 3x + 4$:

a. Compare it to $3\ f(x)$.

b. Compare it to $f(3x)$.

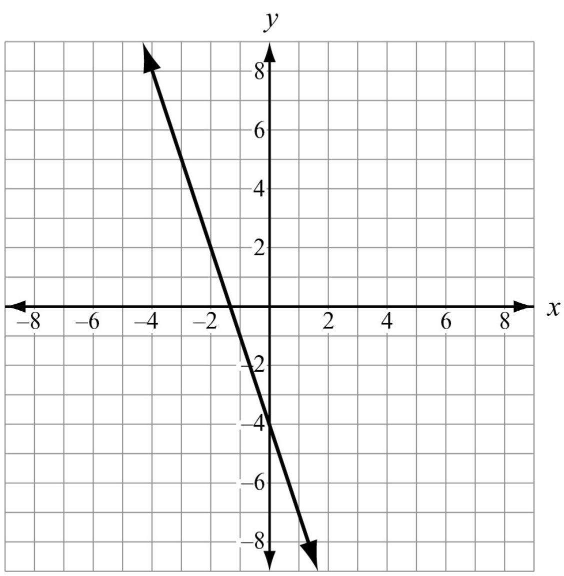
c. Draw a graph of $-f(x)$.

d. Which has the fastest growth rate: $f(x)$, $3f(x)$, or $-f(x)$?

Solution:

a. $3f(x) = 3(3x + 4) = 9x + 12$. So, it crosses the y-axis 8 units higher and has 3 times the growth rate of $f(x)$.

b. $f(3x) = 3(3x) + 4 = 9x + 4$. So, it has the same intercept as $f(x)$ and 3 times the growth rate.

c. 

d. $3f(x)$

EOCT Practice Items

1) A function *g* is an odd function. If *g*(–3) = 4, which of the points lie on the graph of *g*?

- **A.** (3, –4)
- **B.** (–3, –4)
- **C.** (4, –3)
- **D.** (–4, 3)

[Key: A]

2) Which statement is true about the function $f(x) = 7$?

- **A.** The function is odd because $-f(x) = -f(x)$.
- **B.** The function is even because $-f(x) = f(-x)$.
- **C.** The function is odd because $f(x) = f(-x)$.
- **D.** The function is even because $f(x) = f(-x)$.

[Key: D]

CONSTRUCT AND COMPARE LINEAR, ~~QUADRATIC~~, AND EXPONENTIAL MODELS AND SOLVE PROBLEMS

KEY IDEAS

1. Recognizing linear and exponential growth rates is key to modeling a quantitative relationship. The most common growth rates in nature are either linear or exponential. Linear growth happens when the dependent variable changes are the same for equal intervals of the independent variable. Exponential growth happens when the dependent variable changes at the same percent rate for equal intervals of the independent variable.

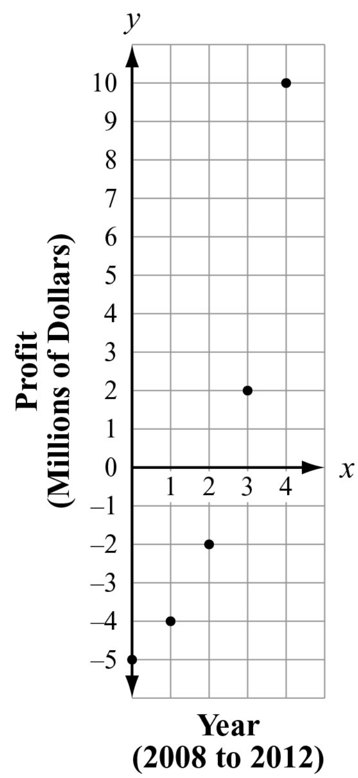
Example:

Given a table of values, look for a constant rate of change in the y, or $f(x)$, column. The table below shows a constant rate of change, namely –2, in the $f(x)$ column for each unit change in the independent variable x. The table also shows the y-intercept of the relation. The function has a y-intercept of +1, the $f(0)$ value. These two pieces of information allow us to find a model for the relationship. When the change in $f(x)$ is constant, we use a linear model, $f(x) = ax + b$, where a represents the constant rate of change and b the y-intercept. For the given table, the a value is –2, the constant change in the $f(x)$ values, and b is the $f(x)$ value of 1. The function is $f(x) = -2x + 1$. Using the linear model, we are looking for an explicit formula for the function.

x	$f(x)$	Change in $f(x)$
–2	5	------------
–1	3	3 – 5 = –2
0	1	1 – 3 = –2
1	3	3 – 1 = –2
2	5	5 – 3 = –2

Example:

Given the graph below, compare the coordinates of points to determine if there is either linear or exponential growth.



The points represent the profit/loss of a new company over its first five years, from 2008 to 2012. The company started out $5,000,000 in debt. After five years it had a profit of $10,000,000. From the arrangement of the points, the pattern does not look linear. We can check by considering the coordinates of the points and using a table of values.

x	*y*	Change in *y*
0	–5,000,000	------------
1	–4,000,000	–4,000,000 – (–5,000,000) = 1,000,000
2	–2,000,000	–2,000,000 – (–4,000,000) = 2,000,000
3	2,000,000	2,000,000 – (–2,000,000) = 4,000,000
4	10,000,000	10,000,000 – (2,000,000) = 8,000,000

The *y* changes are not constant for equal *x* intervals. However, the ratios of successive differences are equal.

$$\frac{2,000,000}{1,000,000} = \frac{4,000,000}{2,000,000} = \frac{2}{1} = 2$$

Having a constant percent for the growth rate for equal intervals indicates exponential growth. The relationship can be modeled using an exponential function. However, our example does not cross the y-axis at 1, or 1,000,000. Since the initial profit value was not 1,000,000, the exponential function has been translated downward. The amount of the translation is $6,000,000. We model the company's growth as:

$$P(x) = 1{,}000{,}000(2^x) - 6{,}000{,}000$$

2. We can use our analysis tools to compare ***growth rates***. For example, it might be interesting to consider whether you would like your pay raises to be linear or exponential. Linear growth is characterized by a constant number. With a linear growth, a value grows by the same amount each time. Exponential growth is characterized by a percent which is called the growth rate.

Example:
Suppose you start work at $600 a week. After a year, you are given two choices for getting a raise: a) 2% a year, or b) a flat $15 a week raise for each successive year. Which option is better? We can make a table with both options and see what happens.

Year	Weekly Pay	
	2% a year	**$15 a week**
0	$600	$600
1	$612	$615
2	$624.26	$630
3	$636.48	$640

Looking at years 1 through 3, the $15 a week option seems better. However, look closely at the 2% column. Though the pay increases start out smaller each year, they are growing exponentially. Some year in the future, the 2% increase in salary will be more than the $15 per week increase in salary. The number of years worked at the company will determine the preferred option.

Important Tips

- Examine function values carefully.
- Remember that a linear function has a constant rate of change.
- Keep in mind that growth rates are modeled with exponential functions.

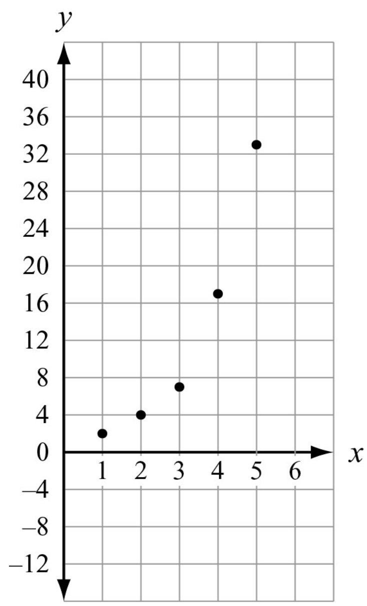
REVIEW EXAMPLES

1) The swans on Elsworth Pond have been increasing in number each year. Felix has been keeping track and so far he has counted 2, 4, 7, 17, and 33 swans each year for the past five years.

   a. Make a scatter plot of the swan population.
   b. What type of model would be a better fit, linear or exponential? Explain your answer.
   c. How many swans should Felix expect next year if the trend continues? Explain your answer.

Solution:

a.



b. Exponential; the growth rate is not constant. The swan population appears to be nearly doubling every year.
c. There could be about 64 swans next year. A function modeling the swan growth would be $P(x) = 2^x$, which would predict $P(6) = 2^6 = 64$.

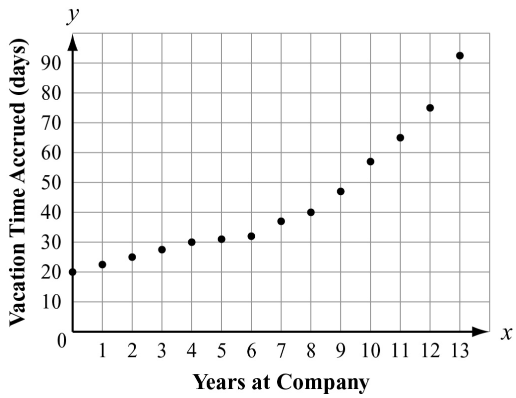
2) Given the sequence 7, 10, 13, 16, . . .
   a. Does it appear to be linear or exponential?
   b. Determine a function to describe the sequence.
   c. What would the 20th term of the sequence be?

Solution:

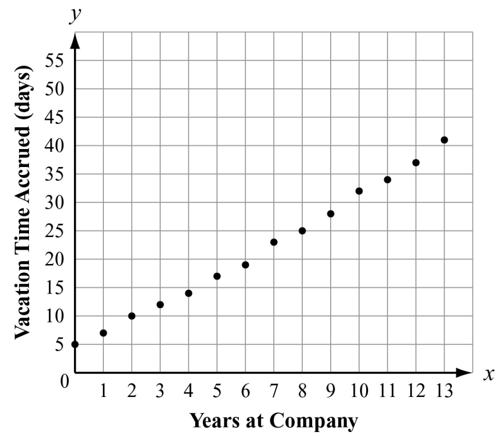
a. Linear; the terms increase by a constant amount, 3.
b. $F(x) = 3x + 4$. The growth rate is 3 and the first term is 4 more than 3 times 1.
c. 83; $F(20) = 3(20) + 4 = 64$

EOCT Practice Items

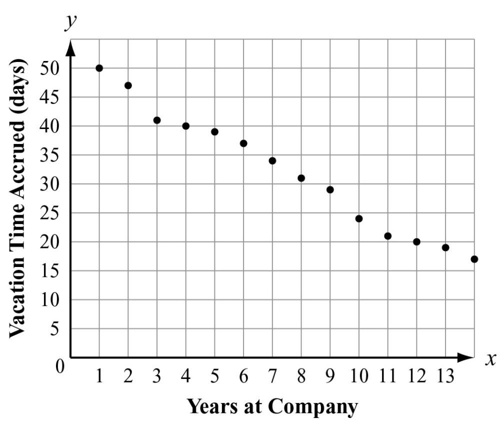
1) Which scatter plot represents a model of linear growth?



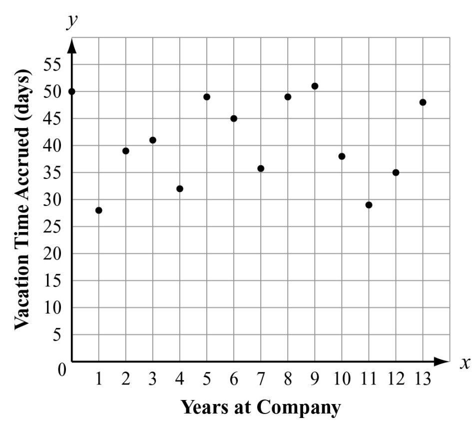
A



B



C

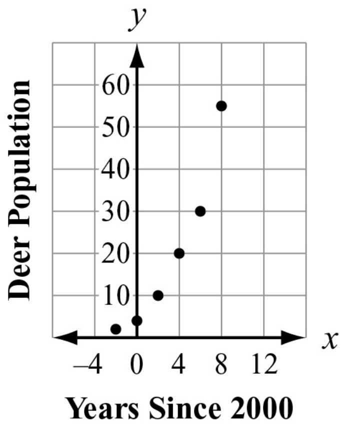


D

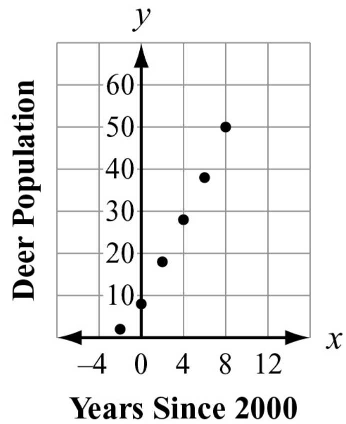
[Key: B]

2) Which scatter plot best represents a model of exponential growth?

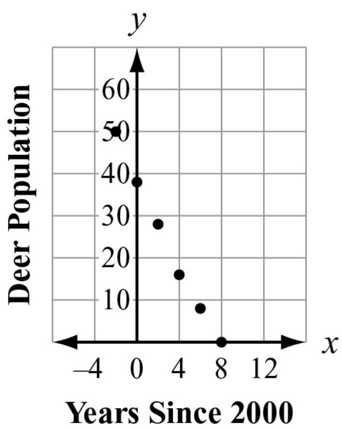
A



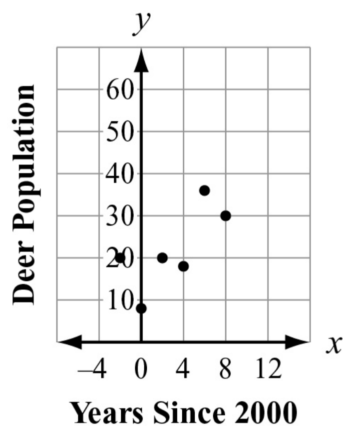
B



C



D



[Key: A]

3) Which table represents a function with a variable growth rate?

A.

x	0	1	2	3	4
y	5	6	7	8	9

B.

x	0	1	2	3	4
y	0	22	44	66	88

C.

x	0	1	2	3	4
y	5	13	21	29	37

D.

x	0	1	2	3	4
y	0	3	9	27	81

[Key: D]

INTERPRET EXPRESSIONS FOR FUNCTIONS IN TERMS OF THE SITUATION THEY MODEL

KEY IDEAS

1. A ***parameter*** is a coefficient or a constant term in the equation that affects the behavior of the function. Though parameters may be expressed as letters when a relationship is generalized, they are not variables. A parameter as a constant term generally affects the intercepts of a function. If the parameter is a coefficient, in general it will affect the rate of change. Below are several examples of specific parameters.

Equation	**Parameter(s)**
$y = 3x + 5$	coefficient 3, constant 5
$f(x) = \frac{9}{5}x + 32$	coefficient $\frac{9}{5}$, constant 32
$v(t) = v_0 + at$	coefficient a, constant v_0
$y = mx + b$	coefficient m, constant b

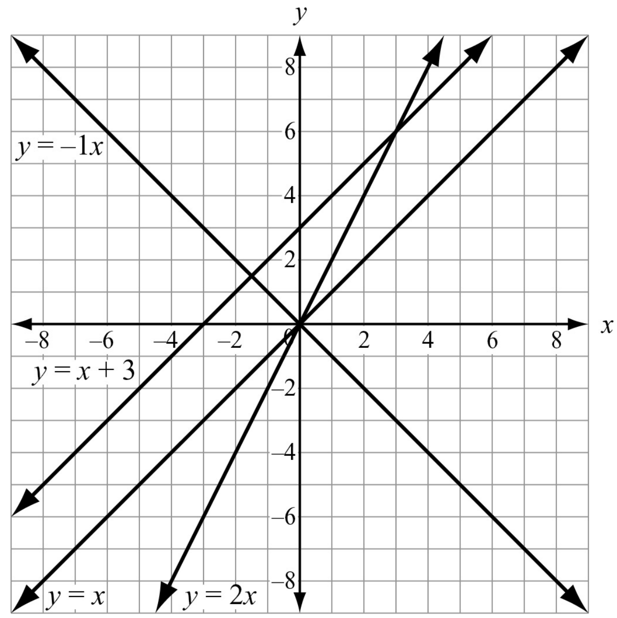
We can look at the effect of parameters on a linear function.

Example:

Consider the lines $y = 1x$, $y = 2x$, $y = -x$, and $y = x + 3$. The coefficients of x are parameters. The +3 in the last equation is a parameter. We can make one table for all four lines and then compare their graphs.

x	***y***			
	$y = 1x$	$y = 2x$	$y = -x$	$y = x + 3$
–3	–3	–6	3	0
–2	–2	–4	2	1
–1	–1	–2	1	2
0	0	0	0	3
1	1	2	–1	4
2	2	4	–2	5
3	3	6	–3	6
4	4	8	-4	7

The four linear graphs show the effects of the parameters.



- Only $y = x + 3$ has a different intercept. The $+3$ translated the $y = x$ graph up 3 units.
- Both $y = x$ and $y = x + 3$ have the same slope (rate of change). The coefficients of the x terms are both 1.
- The lines $y = -x$ and $y = 2x$ have different slopes than $y = x$. The coefficients of the x terms, –1 and 2, affect the lines' slopes.
- The line $y = -x$ is the reflection of $y = x$ over the x-axis. It is the only line with a negative slope.
- The rate of change of $y = 2x$ is twice that of $y = x$.

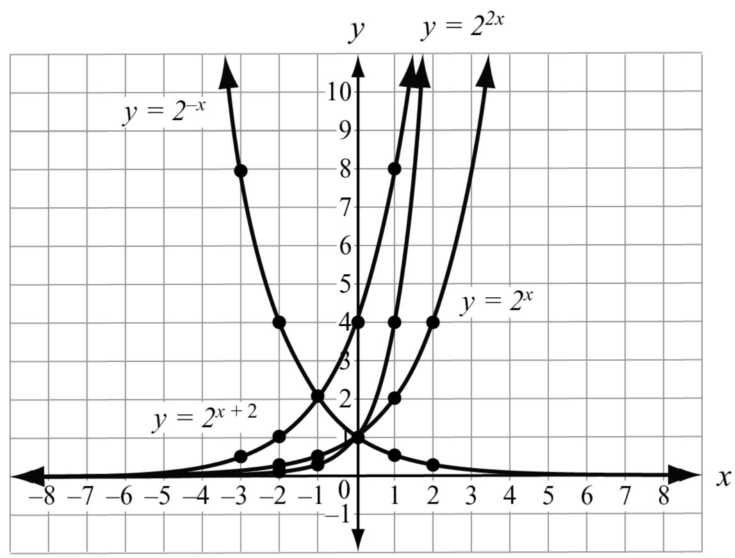
2. We can look at the effect of parameters on an exponential function, in particular when applied to the independent variable, not the base.

Example:

Consider the exponential curves $y = 2^x$, $y = 2^{-x}$, $y = 2^{3x}$, and $y = 2^{x+3}$. The coefficients of the exponent x are parameters. The $+3$ applied to the exponent x in the last equation is a parameter. We can make one table for all four exponentials and then compare the effects.

- $y = 2^{-x}$ is a mirror image of $y = 2^{x}$ with the y-axis as mirror. It has the same y-intercept.
- $y = 2^{2x}$ has the same intercept as $y = 2^{x}$, but rises much more steeply.
- $y = 2^{x+3}$ is the $y = 2^{x}$ curve translated 3 units to the left.

x	y			
	$y = 2^{x}$	$y = 2^{-x}$	$y = 2^{2x}$	$y = 2^{x+3}$
–3	$\frac{1}{8}$	8	2^{-9}	1
–2	$\frac{1}{4}$	4	2^{-6}	2
–1	$\frac{1}{2}$	2	$\frac{1}{8}$	4
0	1	1	1	8
1	2	$\frac{1}{2}$	8	16
2	4	$\frac{1}{4}$	$2^{6} = 64$	32
3	8	$\frac{1}{8}$	$2^{9} = 512$	64
4	16	$\frac{1}{16}$	$2^{12} = 4096$	128



3. Parameters show up in equations when there is a parent or prototype function. Parameters affect the shape and position of the parent function. When we determine a function that models a specific set of data, we are often called upon to find the parent function's parameters.

Example:

Katherine has heard that you can estimate the outside temperature from the number of times a cricket chirps. It turns out that the warmer it is outside the more a cricket will chirp. She has these three pieces of information:

- ✓ cricket chips 76 times a minute at 56° (76, 56)
- ✓ cricket chips 212 times per minute at 90° (212, 90)
- ✓ the relationship is linear

Estimate the function.

Solution:

The basic linear model or parent function is $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept.

So, the slope, or rate of change, is one of our parameters. First we will determine the constant rate of change, called the slope, m.

$$m = \frac{90 - 56}{212 - 76} = \frac{34}{136} = \frac{1}{4}$$

Since we now know that $f(x) = \frac{1}{4}x + b,$ we can substitute in one of our ordered pairs to determine b.

$$T(76) = 56, \text{ so } \frac{1}{4}(76) + b = 56$$

$$19 + b = 56$$

$$19 + b - 19 = 56 - 19$$

$$b = 37$$

Our parameters are $m = \frac{1}{4}$ and $b = 37$.

Our function for the temperature is $T(c) = \frac{1}{4}c + 37.$

REVIEW EXAMPLES

1) Alice finds her flower bulbs multiply each year. She started with just 24 tulip plants. After one year she had 72 plants. Two years later she had 216. Find a linear function to model the growth of Alice's bulbs.

Solution:

The data points are (0, 24), (1, 72), (2, 120). The linear model is $B(p) = m(p) + b$.

We know $b = 24$ because $B(0) = 24$ and $B(0)$ gives the vertical intercept.

Find m:

$m = \frac{120 - 72}{2 - 1} = \frac{48}{1} = 48.$ The parameters are $m = 48$ and $b = 24$.

The function modeling the growth of the bulbs is $B(p) = 48p + 24$.

2) Suppose Alice discovers she counted wrong the second year and she actually had 216 tulip plants. She realizes the growth is not linear because the rate of change was not the same. She must use an exponential model for the growth of her tulip bulbs. Find the exponential function to model the growth.

Solution:

We now have the points (0, 24), (1, 72), (2, 216). We use a parent exponential model:

$$B(p) = a(b^x)$$

In the exponential model the parameter a would be the initial number. So, $a = 24$. To find the base b, we substitute a coordinate pair into the parent function.

$$B(1) = 72\text{, so } 24(b^1) = 72,\ b^1 = 72/24 = 3\text{, so } b = 3.$$

Now we have the parameter and the base. The exponential model for Alice's bulbs would be:

$$\mathrm{B}(p) = 24(3^p).$$

<u>EOCT Practice Item</u>

1) If the parent function is $f(x) = mx + b$, what is the value of the parameter m for the curve passing through the points (–2, 7) and (4, 3)?

A. -9

B. $-\frac{3}{2}$

C. -2

D. $-\frac{2}{3}$

[Key: D]