Unit 5: Transformations in the Coordinate Plane

In this unit, students review the definitions of three types of transformations that preserve distance and angle: rotations, reflections, and translations. They investigate how these transformations are applied in the coordinate plane as functions, mapping pre-image points (inputs) to image points (outputs). Using their knowledge of basic geometric figures and special polygons, they apply these transformations to obtain images of given figures. They also specify transformations that can be applied to obtain a given image from a given pre-image, including cases in which the image and pre-image are the same figure.

KEY STANDARDS

Experiment with transformations in the plane

MCC9-12.G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

MCC9-12.G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

MCC9-12.G.CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

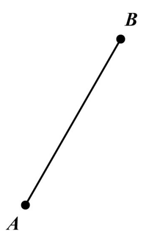
MCC9-12.G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

MCC9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

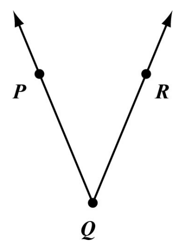
EXPERIMENT WITH TRANSFORMATIONS IN THE PLANE

KEY IDEAS

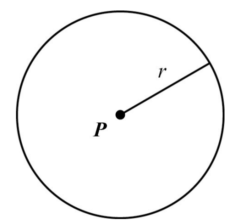
1. A ***line segment*** is part of a line; it consists of two points and all points between them. An ***angle*** is formed by two rays with a common endpoint. A ***circle*** is the set of all points in a plane that are a fixed distant from a given point, called the center; the fixed distance is the ***radius***. ***Parallel lines*** are lines in the same plane that do not intersect. ***Perpendicular lines*** are two lines that intersect to form right angles.



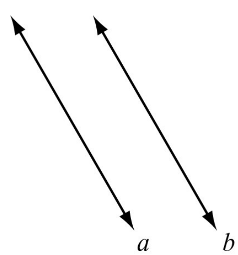
Line segment *AB*



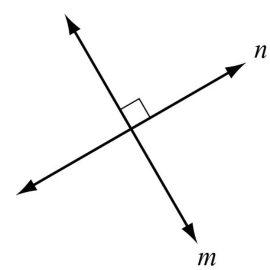
Angle *PQR*



Circle *P* with radius *r*

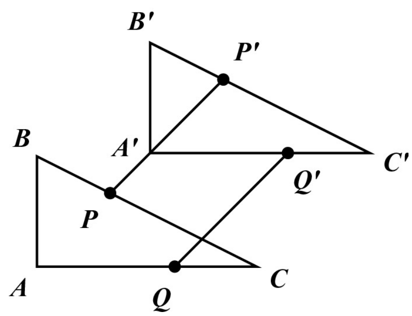


Parallel lines *a* and *b*



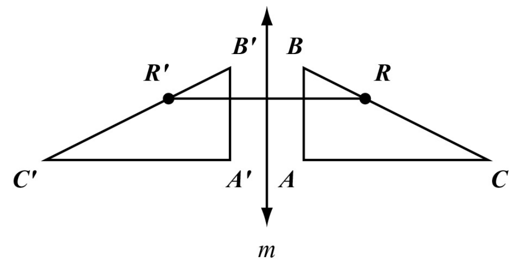
Perpendicular lines *m* and *n*

2. A ***transformation*** is an operation that maps, or moves, a pre-image onto an image. In each transformation defined below, it is assumed that all points and figures are in one plane. In each case, $\triangle ABC$ is the pre-image and $\triangle A'B'C'$ is the image.



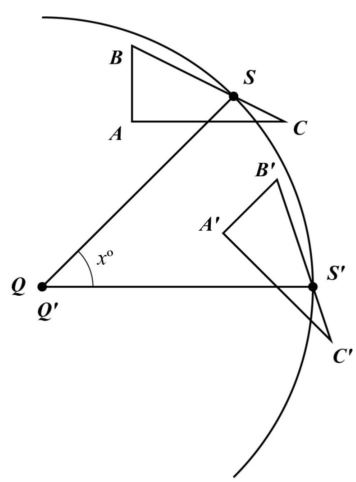
A ***translation*** maps every two points P and Q to points P' and Q' so that the following properties are true:

- $PP' = QQ'$
- $\overline{PP'} \parallel \overline{QQ'}$



A ***reflection*** across a line m maps every point R to R' so that the following properties are true:

- If R is not on m, then m is the perpendicular bisector of $\overline{RR'}$.
- If R is on m, then R and R' are the same point.

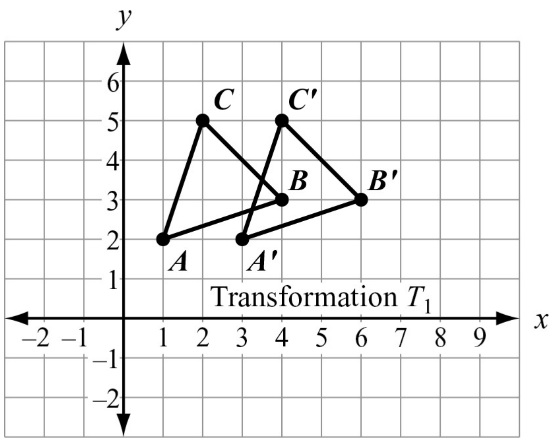


A ***rotation*** of $x°$ about a point Q maps every point S to S' so that the following properties are true:

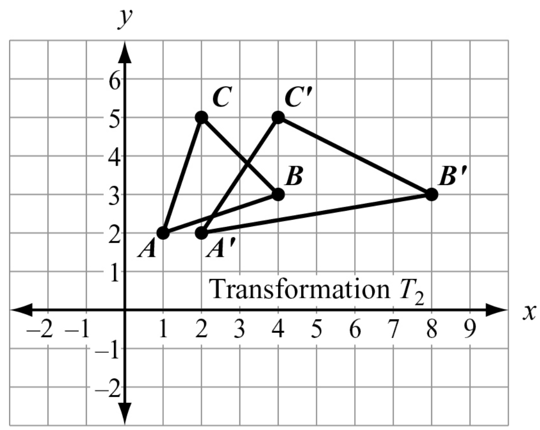
- $SQ = S'Q$ and $m\angle SQS' = x°$.
- Pre-image point Q and image point Q' are the same.

Note: $\overline{QS}$ and $\overline{QS'}$ are radii of a circle with center Q.

3. A transformation in a coordinate plane can be described as a function that maps pre-image points (inputs) to image points (outputs). Translations, reflections, and rotations all preserve distance and angle measure because, for each of those transformations, the pre-image and image are congruent. But some types of transformations do not preserve distance and angle measure because the pre-image and image are not congruent.



T_1: $(x, y) \rightarrow (x + 2, y)$
T_1 translates $\triangle ABC$ to the right 2 units.
T_1 preserves distance and angle measure because $\triangle ABC \cong \triangle A'B'C'$.



T_2: $(x, y) \rightarrow (2x, y)$
T_2 stretches $\triangle ABC$ horizontally by the factor 2.
T_2 preserves neither distance nor angle measure.

4. If vertices are not named, then there might be more than one transformation that will accomplish a specified mapping. If vertices are named, then they must be mapped in a way that corresponds to the order in which they are named.

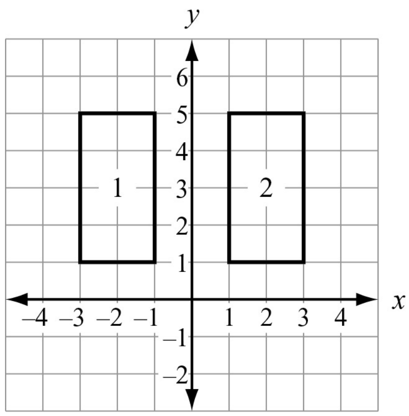
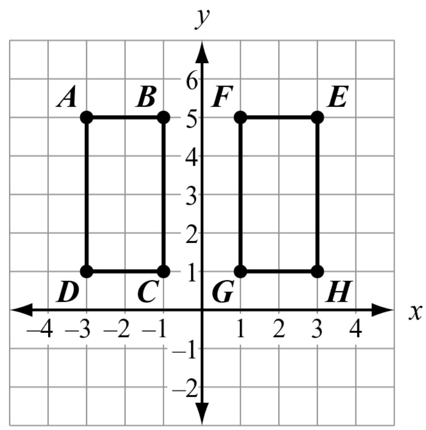


Figure 1 can be mapped to figure 2 by either of these transformations:

- a reflection across the *y*-axis (The upper left vertex in figure 1 is mapped to the upper right vertex in figure 2.), or
- a translation 4 units to the right (The upper left vertex in figure 1 is mapped to the upper left vertex in figure 2.).

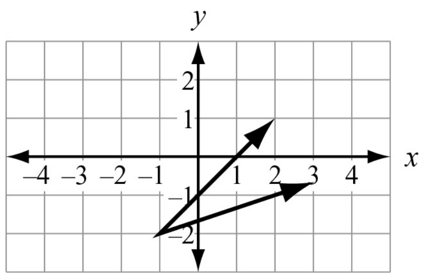


ABCD can be mapped to *EFGH* by a reflection across the *y*-axis, but not by a translation.
The mapping of *ABCD* → *EFGH* requires these vertex mappings:
$A \rightarrow E$, $B \rightarrow F$, $C \rightarrow G$, and $D \rightarrow H$.

REVIEW EXAMPLES

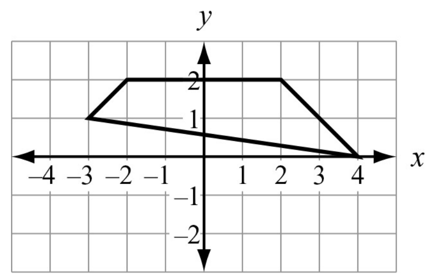
1) Draw the image of each figure, using the given transformation.

a.



Use the translation $(x, y) \rightarrow (x - 3, y + 1)$.

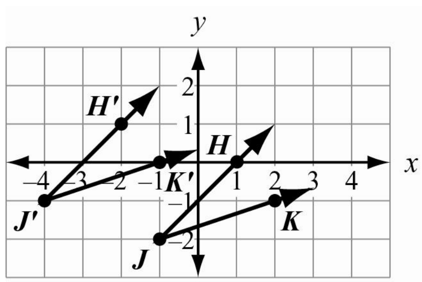
b.



Reflect across the x-axis.

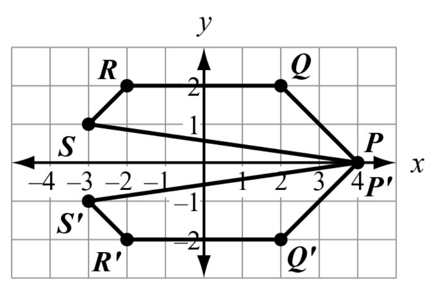
Solution:

a.



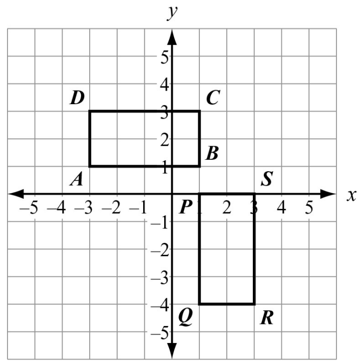
Identify the vertex and a point on each side of the angle. Translate each point 3 units left and 1 unit up. The image of given $\angle HJK$ is $\angle H'J'K'$.

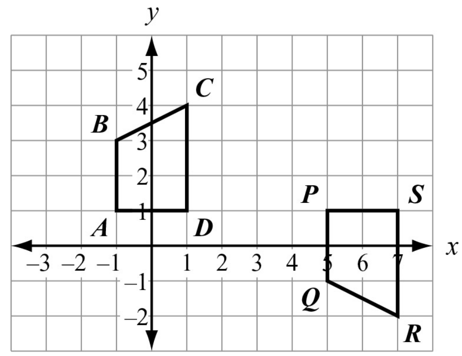
b.



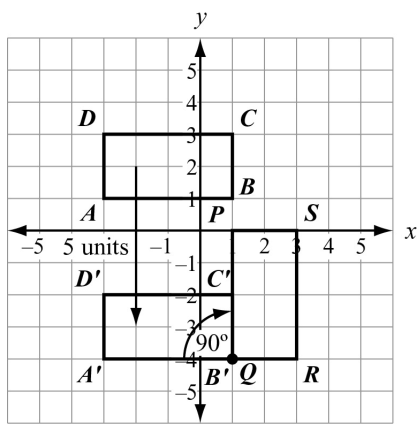
Identify the vertices. The reflection image of each point (x, y) across the x-axis is $(x, -y)$.
The image of given polygon $PQRS$ is $P'Q'R'S'$, where P and P' are the same.

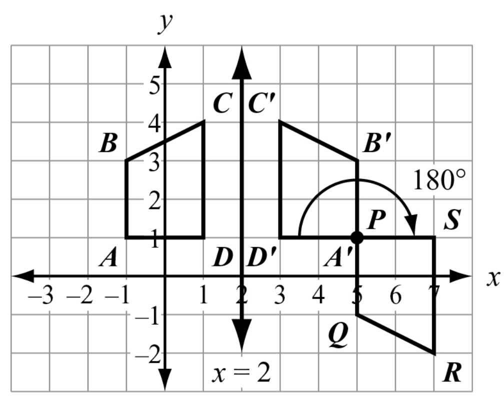
2) Specify a sequence of transformations that will map *ABCD* to *PQRS* in each case.

a. 

b. 

Solution:

a. 

b. 

Translate *ABCD* down 5 units to obtain *A'B'C'D'*. Then rotate *A'B'C'D'* clockwise 90° about point *B'* to obtain *PQRS*.

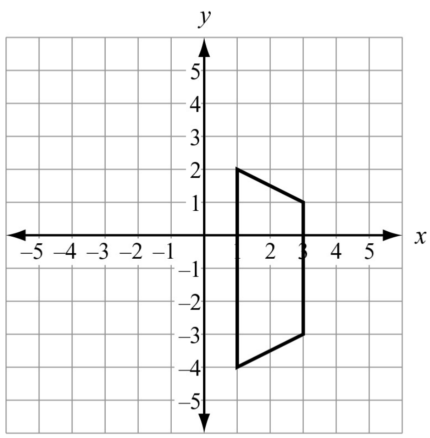
Reflect *ABCD* across the line $x = 2$ to obtain *A'B'C'D'*. Then rotate *A'B'C'D'* 180° about point *A'* to obtain *PQRS*. Note that *A'* and *P* are the same point.

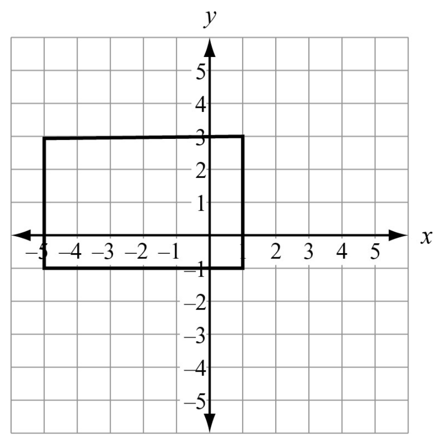
Note that there are other sequences of transformations that will also work for each case.

Important Tip

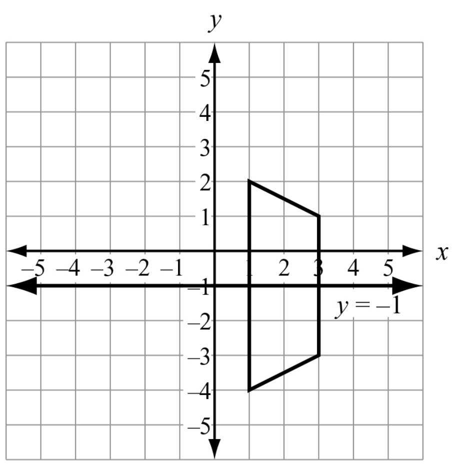
A 180° rotation clockwise is equivalent to a 180° rotation counterclockwise.

3) Describe every transformation that maps the given figure to itself.

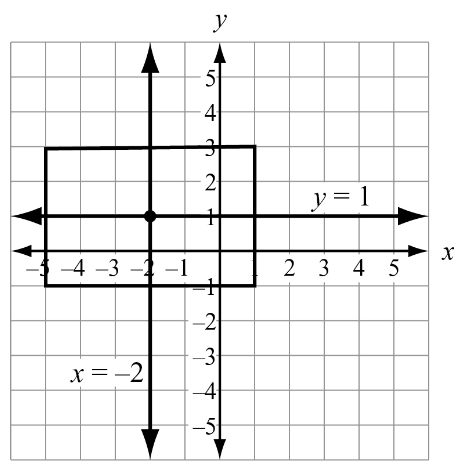
a.


b.


Solution:

a.


There is only one transformation: Reflect the figure across the line $y = -1$.

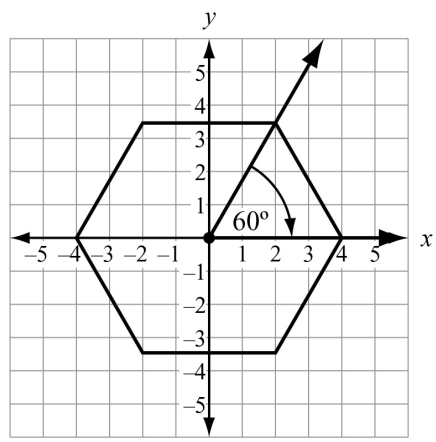
b.


There are three transformations:
- Reflect across the line $y = 1$, or
- Reflect across the line $x = -2$, or
- Rotate 180° about the point (–2, 1).

4) Describe every transformation that maps this figure to itself: a regular hexagon (6 sides) centered about the origin, which has a vertex at (4, 0).

Solution:

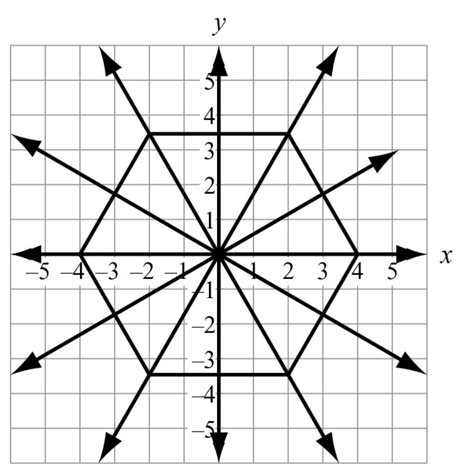
The angle formed by any two consecutive vertices and the center of the hexagon measures 60° because $\frac{360°}{6} = 60°$. So a rotation about the origin, clockwise or counterclockwise, of 60°, 120°, or any other multiple of 60° maps the hexagon to itself.



If a reflection across a line maps a figure to itself, then that line is called a line of symmetry.

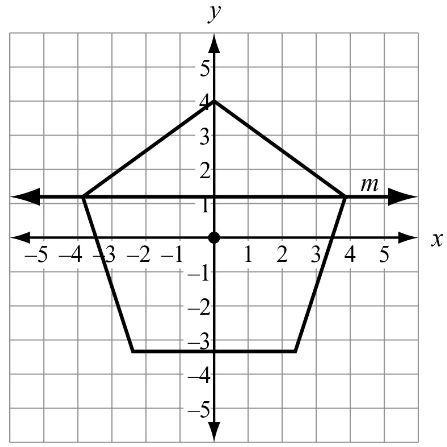
A regular hexagon has 6 lines of symmetry: 3 lines through opposite vertices and 3 lines through midpoints of opposite sides.

A reflection across any of the 6 lines of symmetry maps the hexagon to itself.



EOCT Practice Items

1) A regular pentagon is centered about the origin and has a vertex at (0, 4).

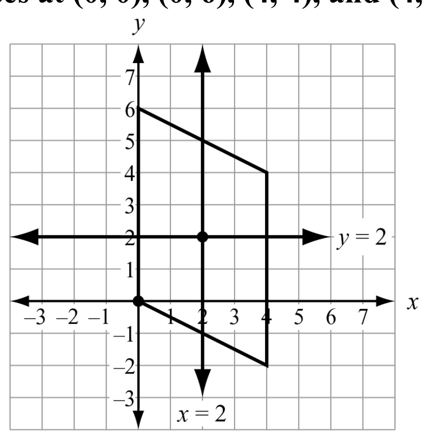


Which transformation maps the pentagon to itself?

A. a reflection across line *m*
B. a reflection across the *x*-axis
C. a clockwise rotation of 100° about the origin
D. a clockwise rotation of 144° about the origin

[Key: D]

2) A parallelogram has vertices at (0, 0), (0, 6), (4, 4), and (4, –2).

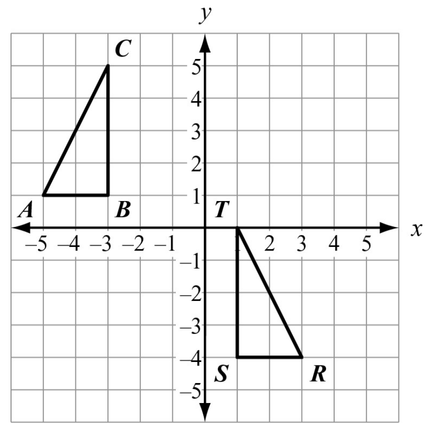


Which transformation maps the parallelogram to itself?

A. a reflection across the line $x = 2$
B. a reflection across the line $y = 2$
C. a rotation of 180° about the point (2, 2)
D. a rotation of 180° about the point (0, 0)

[Key: C]

3) Which sequence of transformations maps $\triangle ABC$ to $\triangle RST$?



A. Reflect $\triangle ABC$ across the line $x = -1$. Then translate the result 1 unit down.
B. Reflect $\triangle ABC$ across the line $x = -1$. Then translate the result 5 units down.
C. Translate $\triangle ABC$ 6 units to the right. Then rotate the result 90° clockwise about the point (1, 1).
D. Translate $\triangle ABC$ 6 units to the right. Then rotate the result 90° counterclockwise about the point (1, 1).

[Key: B]