Unit 6: Connecting Algebra and Geometry Through Coordinates

The focus of this unit is to have students analyze and prove geometric properties by applying algebraic concepts and skills on a coordinate plane. Students learn how to prove the fundamental theorems involving parallel and perpendicular lines and their slopes, applying both geometric and algebraic properties in these proofs. They also learn how to prove other theorems, applying to figures with specified numerical coordinates. (A theorem is any statement that is proved or can be proved. Theorems can be contrasted with postulates, which are statements that are accepted without proof.)

KEY STANDARDS

Use coordinates to prove simple geometric theorems algebraically.

MCC9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. For *example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point* (*0, 2*). (*Restrict contexts that use distance and slope*.)

MCC9-12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MCC9-12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MCC9-12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★

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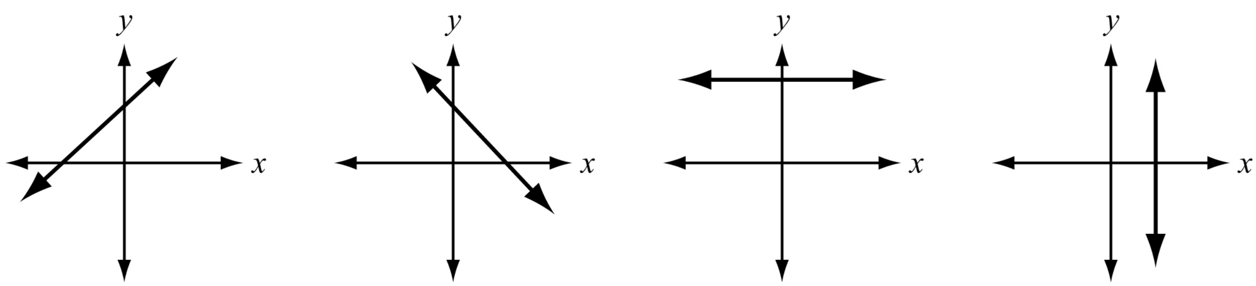
USE COORDINATES TO PROVE SIMPLE GEOMETRIC THEOREMS ALGEBRAICALLY

KEY IDEAS

1. The ***distance*** between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. The ***slope*** of the line through points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.

3. Slopes can be positive, negative, 0, or undefined.



A line with a ***positive slope*** slants up to the right.

A line with a ***negative slope*** slants down to the right.

A line with a ***slope of 0*** is horizontal.

A line with an ***undefined slope*** is vertical.

4. Lines and their slopes are related by the following properties:
   a. Two nonvertical lines are ***parallel*** if and only if they have equal slopes.
   b. Two nonvertical lines are ***perpendicular*** if and only if the product of their slopes is –1.

   Each of these properties has two parts. All parts will be proved in the Review Examples section.

5. Some useful ***properties of proportions*** state that all of the following are equivalent:

$$\frac{a}{b} = \frac{c}{d} \qquad ad = bc \qquad \frac{a}{c} = \frac{b}{d}$$

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Example:

Use the multiplication property of equality to multiply each side of the proportion.

Because $\frac{5}{3}=\frac{10}{6}$, you can also write $5 \bullet 6 = 3 \bullet 10$ and $\frac{5}{10}=\frac{3}{6}$.

Multiply each side by 18, a common multiple of 3 and 6.

$$\frac{5}{3}(18)=(18)\frac{10}{6}$$

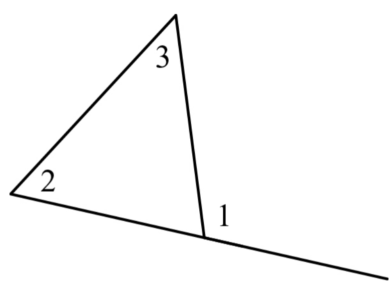
$$5 \bullet 6 = 3 \bullet 10$$

Multiply each side by $\frac{3}{10}$.

$$\frac{3}{10}\bullet\frac{5}{3}=\frac{3}{10}\bullet\frac{10}{6}$$

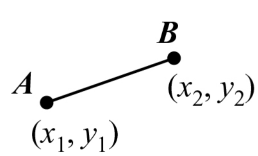
$$\frac{5}{10}=\frac{3}{6}$$

6. ***Exterior Angle Theorem.*** The measure of each exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.



$$m\angle 1 = m\angle 2 + m\angle 3$$

7. A ***directed line segment*** is a line segment from one point to another point in the coordinate plane. The ***components*** of directed line segment $\overline{AB}$ shown below are $(x_2 - x_1,\ y_2 - y_1)$. The components describe the direction and length of the directed line segment.



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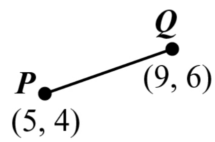
Example:

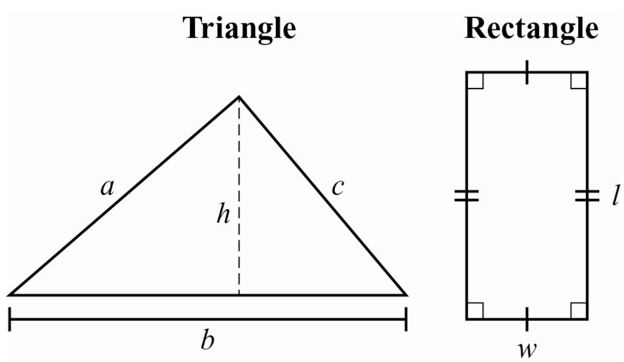
The components of $\overline{PQ}$ are $(9-5,\ 6-4)=(4,\ 2)$. They tell you that a "route" from P to Q is 4 units right and 2 units up. Note that the components are used in the slope: $\frac{6-4}{9-5}=\frac{2}{4}=\frac{1}{2}$.



Important Tip

Directed line segments often represent *vectors*, which are used in advanced math, science, and engineering.

8. The ***perimeter*** of a polygon is the sum of the lengths of the sides. The ***area*** of a polygon is the number of square units enclosed by the polygon.



For a triangle with side lengths a, b, c, with side b as the base and height h, the perimeter P and area A are:

$$P = a + b + c$$

$$A = \frac{1}{2}bh$$

For a rectangle with length l and width w, the perimeter P and area A are:

$$P = 2l + 2w$$

$$A = lw$$

Important Tip

In a triangle, any side can be used as the base. The corresponding height is the altitude

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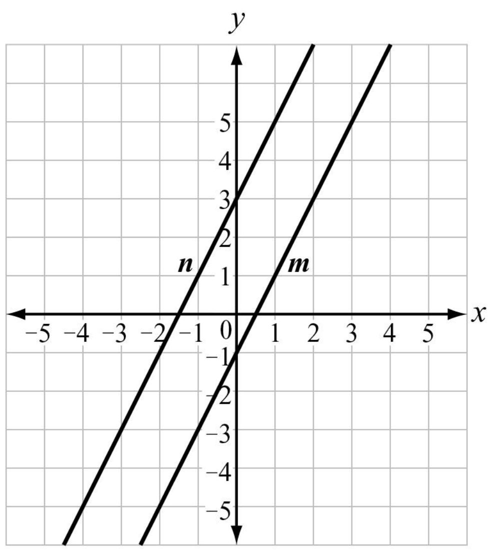
drawn to the line containing that base. In a right triangle, the legs can be used as the base and height.

REVIEW EXAMPLES

1) Prove that if two nonvertical lines are parallel, then they have equal slopes.

Solution:



Lines *n* and *m* are parallel.

Find the slope of line *n*. Two points that lie on line *n* are (1, 5) and (-1, 1).

slope of line $n = \frac{1-5}{-1-1} = \frac{-4}{-2} = 2$

Find the slope of line *m*. Two points that lie on line *m* are (3, 5) and (-1, -3).

slope of line $m = \frac{-3-5}{-1-3} = \frac{-8}{-4} = 2$

So parallel lines *n* and *m* have the same slope.

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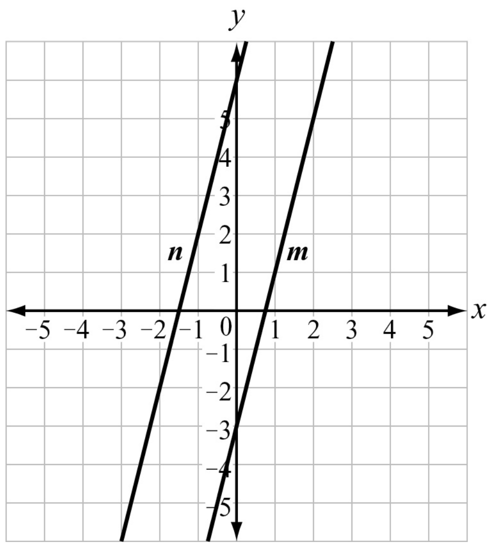
2) Prove that if two nonvertical lines have equal slopes, then they are parallel.

Solution:

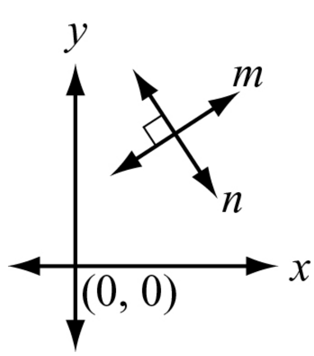


Line n is given by $y = 4x + 6$ and line m is given by $y = 4x - 3$. Both equations are of the form $y = mx + b$. So their slope is 4. By definition, parallel lines have the same slope. So, lines n and m are parallel.

3) Prove that if two nonvertical lines are perpendicular, then the product of their slopes is –1.

Solution:

Given: Slanted lines m and n are perpendicular.
Prove: (slope of m)(slope of n) = –1



Proof:
Name the intersection of lines m and n as point C. Choose any other point A on line n. Draw a horizontal line through A and a vertical line through C; name the intersection of those lines point B and then consider $\triangle ABC$.

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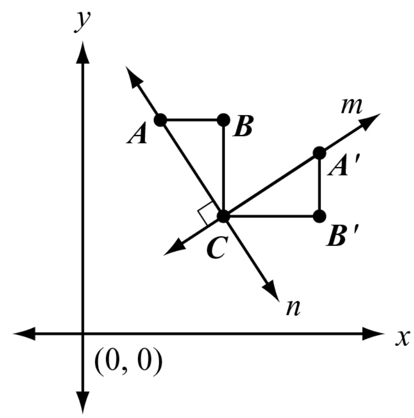
Rotate $\triangle ABC$ clockwise 90° about point C to obtain image $\triangle A'B'C$. Note that $\overline{A'C}$ lies on line m because $m \perp n$, and there is only one line perpendicular to a line through any point on that line.



Rotations preserve distance and angle measure, so $\triangle ABC \cong \triangle A'B'C$.

Because $\triangle ABC \cong \triangle A'B'C$, you know that $AB = A'B'$ and $BC = B'C$.

The slope of line $m = \dfrac{A'B'}{B'C}$.

The slope of line n is negative because line n slants down to the right, so the slope of line $n = -\dfrac{BC}{AB}$.

So (slope of m)(slope of n) = $\left(\dfrac{A'B'}{B'C}\right)\left(-\dfrac{BC}{AB}\right) = -\dfrac{A'B' \cdot BC}{B'C \cdot AB}$.

Finally, substitute AB for $A'B'$ and BC for $B'C$ and then divide out common factors to obtain the statement you need to prove:

(slope of m)(slope of n) = $-\dfrac{AB \cdot BC}{BC \cdot AB} = -\dfrac{\cancel{AB}^{1} \cdot \cancel{BC}^{1}}{\cancel{BC}_{1} \cdot \cancel{AB}_{1}} = -1$.

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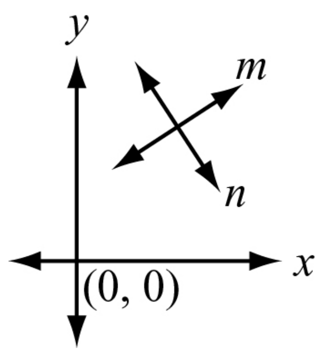
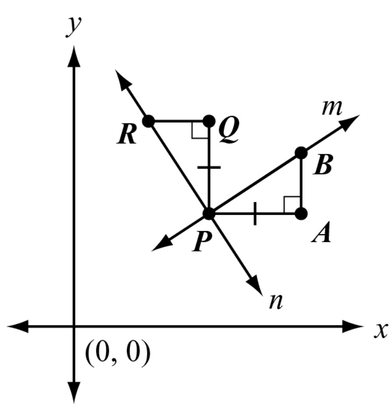
4) Prove that if the product of the slopes of two nonvertical lines is –1, then those lines are perpendicular.

Solution:

Given: (slope of *m*)(slope of *n*) = –1
Prove: $m \perp n$



Proof:
Name the intersection of lines *m* and *n* point *P*.
Choose any other point *R* on line *n*.
Draw a horizontal line through *R* and a vertical line through *P*; name the intersection of those lines point *Q* and then consider $\triangle PQR$. Note that $\angle Q$ is a right angle because it is formed by a horizontal line and a vertical line.



Draw a horizontal line through *P* and mark a point *A* on that line to make *PA* = *PQ*.
Draw a vertical line through *A*; name the intersection of that line and line *m* point *B* and then consider $\triangle PAB$. Note that $\angle A$ is a right angle because it is formed by a horizontal line and a vertical line.

Now show that *AB* = *QR*:

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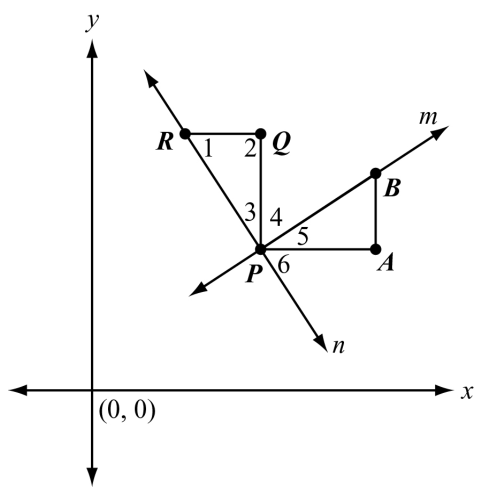
(slope of m)(slope of n) $= -1$	Given
$\left(\frac{AB}{PA}\right)\left(-\frac{PQ}{QR}\right) = -1$	Definition of slope
$\left(\frac{AB}{PA}\right)\left(\frac{PQ}{QR}\right) = 1$	Multiply both sides by -1.
$\left(\frac{AB}{PA}\right)\left(\frac{PQ}{QR}\right)\left(\frac{QR}{PQ}\right) = 1\left(\frac{QR}{PQ}\right)$	Multiply both sides by $\frac{QR}{PQ}$.
$\left(\frac{AB}{PA}\right)\left(\frac{\cancel{PQ}}{\cancel{QR}}\right)\left(\frac{\cancel{QR}}{\cancel{PQ}}\right) = 1\left(\frac{QR}{PQ}\right)$	Divide out common factors.
$\left(\frac{AB}{PA}\right) = \left(\frac{QR}{PQ}\right)$	Simplify.
$\left(\frac{AB}{PA}\right) = \left(\frac{QR}{PA}\right)$	Substitute PA for PQ.
$AB = QR$	Multiply both sides by PA and simplify.

Now consider the numbered angles.



The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles, so:

$$\underbrace{m\angle 4 + m\angle 5 + m\angle 6}_{\substack{\uparrow \\ \text{exterior angle of} \\ \triangle PQR}} = \underbrace{m\angle 1 + m\angle 2}_{\substack{\uparrow \\ \text{two remote interior} \\ \text{angles of } \triangle PQR}}$$

But the triangles are congruent, so $m\angle 3 = m\angle 5$. Substitute $m\angle 3$ for $m\angle 5$ in the above equation:

$$m\angle 4 + m\angle 3 + m\angle 6 = m\angle 1 + m\angle 2$$

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Now note that parallel line segments $\overline{RQ}$ and $\overline{PA}$ are intersected by the transversal *n*, so $m\angle 6 = m\angle 1$ (corresponding angles).

Subtract those equal measures:

$$\begin{aligned} m\angle 4 + m\angle 3 + m\angle 6 &= m\angle 1 + m\angle 2 \\ \underline{-\,m\angle 6} &= \underline{-\,m\angle 1} \\ m\angle 4 + m\angle 3 &= m\angle 2 \end{aligned}$$

So $m\angle 4 + m\angle 3 = m\angle 2 = 90°$, which proves that $m \perp n$.

5) The line *p* is represented by the equation $y = 4x + 1$. What is the equation of the line that is perpendicular to line *p* and passes through the point (8, 5)?

Solution:

Identify the slope of the line perpendicular to line *p*. The slopes of perpendicular lines are negative reciprocals. Since the slope of line *p* is 4, $m = -\frac{1}{4}$.

The slope-intercept form of the equation of a line is $y = mx + b$. Substitute $-\frac{1}{4}$ for *m*. The line perpendicular to line *p* passes through (8, 5), so substitute 8 for *x* and 5 for *y*. Solve for *b*.

$5 = -\frac{1}{4}(8) + b$

$5 = -2 + b$

$7 = b$

The equation of the line perpendicular to line *p* and that passes through (8, 5) is $y = -\frac{1}{4}$.

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6) Prove or disprove that the point $\left(1, \sqrt{3}\right)$ lies on the circle centered at the origin and containing the point (0, 2).

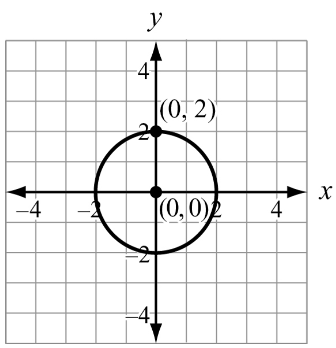
Solution:

The point (0, 2) is 2 units from the origin (0, 0), so the circle centered at the origin and containing the point (0, 2) has a radius of 2.

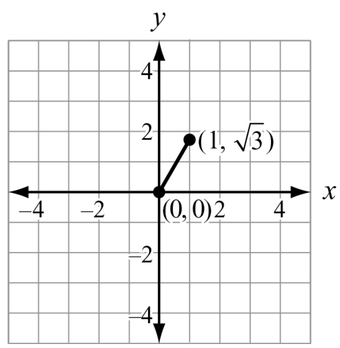
A point is on that circle if and only if it is 2 units from (0, 0).

The distance from $\left(1, \sqrt{3}\right)$ to (0, 0) is: $\sqrt{(1-0)^2 + \left(\sqrt{3}-0\right)^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$.

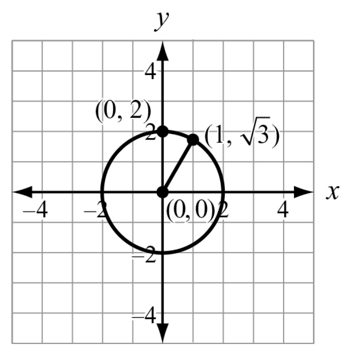
Therefore, the point $\left(1, \sqrt{3}\right)$ does lie on the circle centered at the origin and containing the point (0, 2).



The circle centered at (0, 0) and containing the point (0, 2)



This distance from $\left(1, \sqrt{3}\right)$ to (0, 0)



The point $\left(1, \sqrt{3}\right)$ is on the circle

7) Prove that *ABCD* is a rectangle if the vertices are *A*(4, 0), *B*(3, 3), *C*(–3, 1), and *D*(–2, –2).

Solution:

The slopes of the sides are:

$\overline{AB}: \dfrac{3-0}{3-4} = \dfrac{3}{-1} = -3$ $\qquad \overline{BC}: \dfrac{1-3}{-3-3} = \dfrac{-2}{-6} = \dfrac{1}{3}$

$\overline{CD}: \dfrac{-2-1}{-2+3} = \dfrac{-3}{1} = -3$ $\qquad \overline{DA}: \dfrac{0+2}{4+2} = \dfrac{2}{6} = \dfrac{1}{3}$

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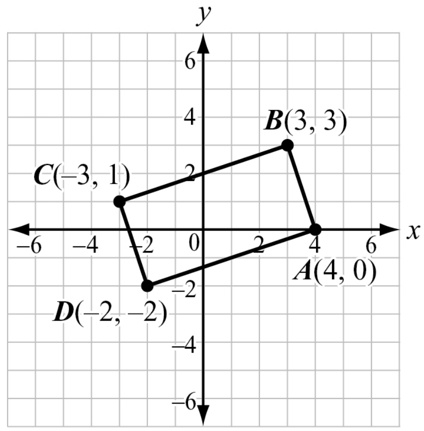


$\overline{AB} \parallel \overline{CD}$ because they have equal slopes.

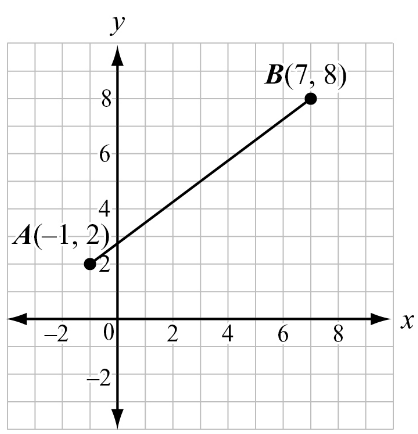
$\overline{BC} \parallel \overline{DA}$ because they have equal slopes.

So *ABCD* is a parallelogram because both pairs of opposite sides are parallel.

$\overline{AB} \perp \overline{BC}$ because the product of their slopes is –1: $-3 \cdot \frac{1}{3} = -1$.

Therefore, *ABCD* is a rectangle because it is a parallelogram with a right angle.

8) Given the points *A*(–1, 2) and *B*(7, 8), find the coordinates of the point *P* on directed line segment $\overline{AB}$ that partitions $\overline{AB}$ in the ratio $\frac{1}{3}$.



Solution:

Point *P* partitions $\overline{AB}$ in the ratio $\frac{1}{3}$ if *P* is on $\overline{AB}$ and $\overline{AP} = \frac{1}{3}\overline{PB}$.

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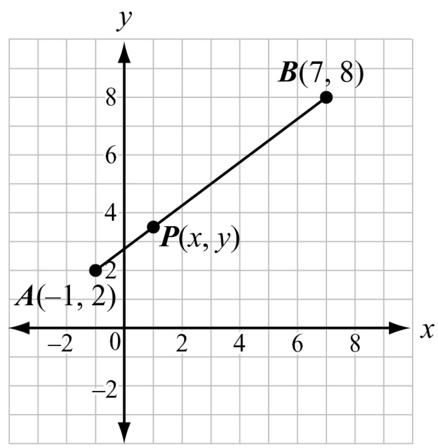

Let P be on $\overline{AB}$ and have coordinates (x, y). Use components and solve two equations to find x and y:

$$\overline{AP} = \frac{1}{3}\overline{PB}$$

$$(x+1,\ y-2) = \frac{1}{3}(7-x,\ 8-y)$$

$$\begin{aligned} x+1 &= \frac{1}{3}(7-x) \\ 3 \cdot (x+1) &= 3 \cdot \frac{1}{3}(7-x) \\ 3x+3 &= 7-x \\ 4x+3 &= 7 \\ 4x &= 4 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} y-2 &= \frac{1}{3}(8-y) \\ 3 \cdot (y-2) &= 3 \cdot \frac{1}{3}(8-y) \\ 3y-6 &= 8-y \\ 4y-6 &= 8 \\ 4y &= 14 \\ y &= \frac{14}{4} \\ y &= \frac{7}{2} \end{aligned}$$

The coordinates of P are $\left(1, \frac{7}{2}\right)$.

Important Tip

$\overline{AB}$ and $\overline{BA}$ have opposite components. The point Q that partitions $\overline{BA}$ in the ratio $\frac{1}{3}$ is $\left(5, \frac{13}{2}\right)$. The first step in finding these coordinates is shown below.

$$\overline{BQ} = \frac{1}{3}\overline{QA}$$

$$(x-7,\ y-8) = \frac{1}{3}(-1-x,\ 2-y)$$

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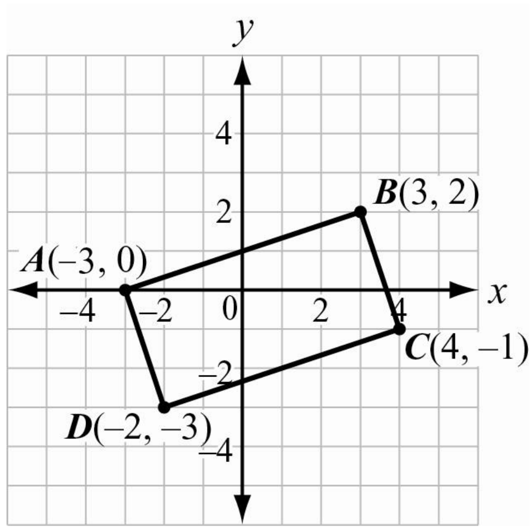
9) Find the perimeter and area of rectangle $ABCD$ with vertices $A(-3, 0)$, $B(3, 2)$, $C(4, -1)$, and $D(-2, -3)$.

Solution:



Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the length and width of the rectangle.

$AB = \sqrt{(3-(-3))^2 + (2-0)^2} = \sqrt{36+4} = \sqrt{40}$

$BC = \sqrt{(4-3)^2 + (-1-2)^2} = \sqrt{1+9} = \sqrt{10}$

The length of the rectangle is usually considered to be the longer side. Therefore, the length of the rectangle is $2\sqrt{10}$ and the width is $\sqrt{10}$.

Use formulas to find the perimeter and area of the rectangle**.**

$P = 2l + 2w$

$P = 2(2\sqrt{10}) + 2(\sqrt{10})$

$P = 6\sqrt{10}$

The perimeter of the rectangle is $6\sqrt{10}$ units.

$A = lw$

$A = (2\sqrt{10})(\sqrt{10})$

$A = 20$

The area of the rectangle is 20 square units.

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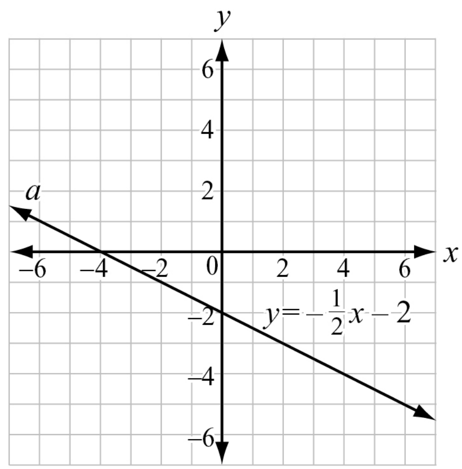
EOCT Practice Items

1) An equation of line *a* is $y = -\frac{1}{2}x - 2$.



Which is an equation of the line that is perpendicular to line *a* and passes through the point (–4, 0)?

A. $y = -\frac{1}{2}x + 2$

B. $y = -\frac{1}{2}x + 8$

C. $y = 2x - 2$

D. $y = 2x + 8$

[Key: D]

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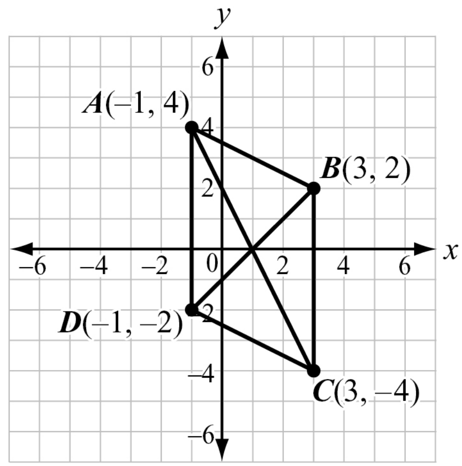
2) Parallelogram *ABCD* has vertices as shown.



Which equation would be used in proving that the diagonals of parallelogram *ABCD* bisect each other?

A. $\sqrt{(3-1)^2+(2-0)^2}=\sqrt{(1-3)^2+(0+4)^2}$

B. $\sqrt{(3+1)^2+(2+0)^2}=\sqrt{(1+3)^2+(0-4)^2}$

C. $\sqrt{(-1-1)^2+(4-0)^2}=\sqrt{(1-3)^2+(0+4)^2}$

D. $\sqrt{(-1+1)^2+(4+0)^2}=\sqrt{(1+3)^2+(0-4)^2}$

[Key: C]

3) Given the points *P*(2, –1) and *Q*(–9, –6), what are the coordinates of the point on directed line segment $\overline{PQ}$ that partitions $\overline{PQ}$ in the ratio $\frac{3}{2}$?

A. $\left(-\frac{23}{5},-4\right)$

B. $\left(-\frac{12}{5},-3\right)$

C. $\left(-\frac{5}{3},-\frac{8}{3}\right)$

D. $\left(-\frac{5}{3},-\frac{8}{3}\right)$

[Key: A]

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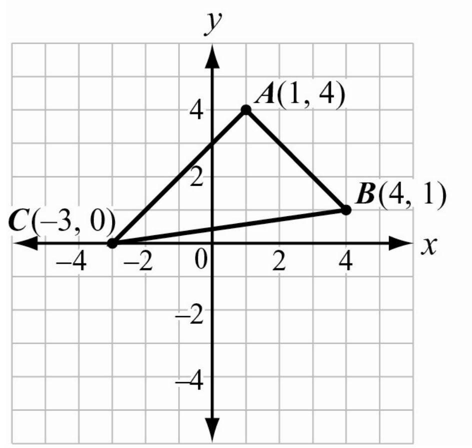
4) Triangle *ABC* has vertices as shown.



What is the area of the triangle?

A. $\sqrt{72}$ square units
B. 12 square units
C. $\sqrt{288}$ square units
D. 24 square units

[Key: B]

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