EXPLORING DATA AND STATISTICS

11.5

Recursive Rules for Sequences

GOAL 1 USING RECURSIVE RULES FOR SEQUENCES

So far in this chapter you have worked with *explicit rules* for the *n*th term of a sequence, such as $a_n = 3n - 2$ and $a_n = 3(2)^n$. An **explicit rule** gives a_n as a function of the term's position number n in the sequence.

In this lesson you will learn another way to define a sequence—by a *recursive rule*. A **recursive rule** gives the beginning term or terms of a sequence and then a *recursive equation* that tells how a_n is related to one or more preceding terms.

What you should learn

GOAL 1 Evaluate and write recursive rules for sequences.

GOAL 2 Use recursive rules to solve **real-life** problems, such as finding the number of fish in a lake in **Example 5**.

Why you should learn it

▼ To model **real-life** quantities, such as the number of trees on a tree farm in **Exs. 49 and 50**.

REAL LIFE

EXAMPLE 1 *Evaluating Recursive Rules*

Write the first five terms of the sequence.

a. *Factorial numbers*: $a_0 = 1, a_n = n \cdot a_{n-1}$

b. *Fibonacci sequence*: $a_1 = 1, a_2 = 1, a_n = a_{n-2} + a_{n-1}$

SOLUTION

a. $a_0 = 1$

$a_1 = 1 \cdot a_0 = 1 \cdot 1 = 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2$

$a_3 = 3 \cdot a_2 = 3 \cdot 2 = 6$

$a_4 = 4 \cdot a_3 = 4 \cdot 6 = 24$

b. $a_1 = 1$

$a_2 = 1$

$a_3 = a_1 + a_2 = 1 + 1 = 2$

$a_4 = a_2 + a_3 = 1 + 2 = 3$

$a_5 = a_3 + a_4 = 2 + 3 = 5$

The factorial numbers in part (a) of Example 1 are denoted by a special symbol, !, called a **factorial** symbol. The expression $n!$ is read "*n* factorial" and represents the product of all integers from 1 to n. Here are several factorial values.

$0! = 1$ (by definition) $\quad 1! = 1 \quad 2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6 \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \quad 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

ACTIVITY Developing Concepts — Investigating Recursive Rules

1. Find the first five terms of each sequence.

   a. $a_1 = 3$
   $a_n = a_{n-1} + 5$

   b. $a_1 = 3$
   $a_n = 2a_{n-1}$

2. Based on the lists of terms you found in **Step 1**, what type of sequence is the sequence in part (a)? in part (b)?

EXAMPLE 2 Writing a Recursive Rule for an Arithmetic Sequence

Write the indicated rule for the arithmetic sequence with $a_1 = 4$ and $d = 3$.

a. an explicit rule **b.** a recursive rule

SOLUTION

a. From Lesson 11.2 you know that an explicit rule for the nth term of the arithmetic sequence is:

$a_n = a_1 + (n - 1)d$ General explicit rule for a_n

$= 4 + (n - 1)3$ Substitute for a_1 and d.

$= 1 + 3n$ Simplify.

b. To find the recursive equation, use the fact that you can obtain a_n by adding the common difference d to the previous term.

$a_n = a_{n-1} + d$ General recursive rule for a_n

$= a_{n-1} + 3$ Substitute for d.

A recursive rule for the sequence is $a_1 = 4$, $a_n = a_{n-1} + 3$.

EXAMPLE 3 Writing a Recursive Rule for a Geometric Sequence

Write the indicated rule for the geometric sequence with $a_1 = 3$ and $r = 0.1$.

a. an explicit rule **b.** a recursive rule

SOLUTION

a. From Lesson 11.3 you know that an explicit rule for the nth term of the geometric sequence is:

$a_n = a_1 r^{n-1}$ General explicit rule for a_n

$= 3(0.1)^{n-1}$ Substitute for a_1 and r.

b. To write a recursive rule, use the fact that you can obtain a_n by multiplying the previous term by r.

$a_n = r \cdot a_{n-1}$ General recursive rule for a_n

$= (0.1)a_{n-1}$ Substitute for r.

A recursive rule for the sequence is $a_1 = 3$, $a_n = (0.1)a_{n-1}$.

EXAMPLE 4 Writing a Recursive Rule

Write a recursive rule for the sequence 1, 2, 2, 4, 8, 32, ...

SOLUTION

Beginning with the third term in the sequence, each term is the product of the two previous terms. Therefore, a recursive rule is given by:

$$a_1 = 1, a_2 = 2, a_n = a_{n-2} \cdot a_{n-1}$$

FOCUS ON CAREERS

FISHERY BIOLOGIST

Some fishery biologists manage fish hatcheries, conduct fish disease control programs, and work with organizations to restore and enhance fish habitats.

CAREER LINK
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GOAL 2 USING RECURSIVE RULES IN REAL LIFE

EXAMPLE 5 *Using a Recursive Rule*

FISH A lake initially contains 5200 fish. Each year the population declines 30% due to fishing and other causes, and the lake is restocked with 400 fish.

a. Write a recursive rule for the number a_n of fish at the beginning of the nth year. How many fish are in the lake at the beginning of the fifth year?

b. What happens to the population of fish in the lake over time?

SOLUTION

a. Because the population declines 30% each year, 70% of the fish remain in the lake from one year to the next, and new fish are added.

VERBAL MODEL

Fish at start of nth year $=$ **0.7** **Fish at start of $(n - 1)$st year** $+$ **New fish added**

LABELS

Fish at start of nth year $= a_n$

Fish at start of $(n - 1)$st year $= a_{n-1}$

New fish added $=$ **400**

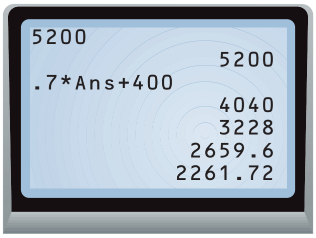
ALGEBRAIC MODEL

$$a_n = (0.7)\,a_{n-1} + 400$$

A recursive rule is:

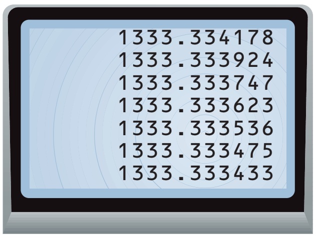
$a_1 = 5200,\ a_n = (0.7)a_{n-1} + 400$

You can use a graphing calculator to find a_5, the number of fish in the lake at the beginning of the fifth year. Enter the number of fish at the beginning of the first year, which is $a_1 = 5200$. Then enter the rule $0.7 \times \text{Ans} + 400$ to find a_2. Press ENTER three more times to find $a_5 \approx 2262$.



▶ There are about 2262 fish in the lake at the beginning of the fifth year.

b. To determine what happens to the lake's fish population over time, continue pressing ENTER on the calculator. The calculator screen at the right shows the fish populations for years 44–50. Observe that the numbers approach about 1333.



▶ Over time, the population of fish in the lake stabilizes at about 1333 fish.

Guided Practice

Vocabulary Check ✓

1. Complete this statement: The expression __?__ represents the product of all integers from 1 to *n*.

Concept Check ✓

2. Explain the difference between an explicit rule for a sequence and a recursive rule for a sequence.

3. Give an example of an explicit rule for a sequence and a recursive rule for a sequence.

Skill Check ✓

Write the first five terms of the sequence.

4. $a_1 = 1$
$a_n = a_{n-1} + 1$

5. $a_1 = 2$
$a_n = 4a_{n-1}$

6. $a_0 = 1$
$a_n = a_{n-1} - 2$

7. $a_1 = -1$
$a_n = -3a_{n-1}$

8. $a_1 = 2$
$a_n = 2a_{n-1} - 3$

9. $a_0 = 3$
$a_n = (a_{n-1})^2 + 1$

Write a recursive rule for the sequence.

10. 21, 17, 13, 9, 5, . . .

11. 2, 6, 18, 54, 162, . . .

12. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

13. **FISH** Suppose each year the lake in Example 5 is restocked with 750 fish. How many fish are in the lake at the beginning of the fifth year?

Practice and Applications

STUDENT HELP
Extra Practice to help you master skills is on p. 956.

WRITING TERMS **Write the first five terms of the sequence.**

14. $a_0 = 1$
$a_n = a_{n-1} + 4$

15. $a_1 = 4$
$a_n = n + a_{n-1} + 6$

16. $a_0 = 0$
$a_n = a_{n-1} - n^2$

17. $a_0 = -4$
$a_n = a_{n-1} - 8$

18. $a_1 = 2$
$a_n = (a_{n-1})^2 + 2$

19. $a_0 = 5$
$a_n = n^2 - a_{n-1}$

20. $a_1 = 10$
$a_n = 3a_{n-1}$

21. $a_0 = 2$
$a_n = n^2 + 2n - a_{n-1}$

22. $a_0 = 3$
$a_n = (a_{n-1})^2 - 2$

23. $a_0 = 48$
$a_n = \frac{1}{2}a_{n-1} + 2$

24. $a_0 = 4, a_1 = 2$
$a_n = a_{n-1} - a_{n-2}$

25. $a_1 = 1, a_2 = 3$
$a_n = a_{n-1} \cdot a_{n-2}$

WRITING RULES **Write an explicit rule and a recursive rule for the sequence. (Recall that *d* is the common difference of an arithmetic sequence and *r* is the common ratio of a geometric sequence.)**

STUDENT HELP
HOMEWORK HELP
Example 1: Exs. 14–25
Examples 2, 3: Exs. 26–34
Example 4: Exs. 35–43
Example 5: Exs. 44–54

26. $a_1 = 2$
$r = 10$

27. $a_1 = 3$
$d = 10$

28. $a_1 = 10$
$r = 2$

29. $a_1 = 5$
$d = 3$

30. $a_1 = 0$
$d = -1$

31. $a_1 = 5$
$r = 2.5$

32. $a_1 = 14$
$d = \frac{1}{2}$

33. $a_1 = \frac{1}{2}$
$r = 4$

34. $a_1 = -1$
$d = -\frac{3}{2}$

STUDENT HELP

INTERNET HOMEWORK HELP
Visit our Web site www.mcdougallittell.com for help with Exs. 35–43.

WRITING RULES **Write a recursive rule for the sequence. The sequence may be arithmetic, geometric, or neither.**

35. 1, 7, 13, 19, . . .

36. 66, 33, 16.5, 8.25, . . .

37. 41, 32, 23, 14, . . .

38. 3, 8, 63, 3968, . . .

39. 33, 11, $\frac{11}{3}$, $\frac{11}{9}$, . . .

40. 7.2, 3.2, −0.8, −4.8, . . .

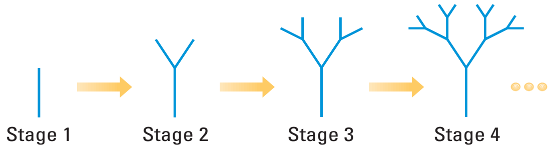
41. 2, 5, 10, 50, 500, . . .

42. 6, $6\sqrt{2}$, 12, $12\sqrt{2}$, . . .

43. 48, 4.8, 0.48, 0.048, . . .

44. **ON LAYAWAY** Suppose you buy a $500 camcorder on layaway by making a down payment of $150 and then paying $25 per month. Write a recursive rule for the total amount of money paid on the camcorder at the beginning of the *n*th month. How much will you have left to pay on the camcorder at the beginning of the twelfth month?

FRACTAL TREE **In Exercises 45 and 46, use the following information.**
A fractal tree starts with a single branch (the trunk). At each stage the new branches from the previous stage each grow two more branches, as shown.



45. List the number of new branches in each of the first seven stages. What type of sequence do these numbers form?

46. Write an explicit rule and a recursive rule for the sequence in Exercise 45.

POOL CARE **In Exercises 47 and 48, use the following information.**
You have just bought a new swimming pool and need to add chlorine to the water. You add 32 ounces of chlorine the first week and 14 ounces every week thereafter. Each week 40% of the chlorine in the pool evaporates.

47. Write a recursive rule for the amount of chlorine in the pool each week. How much chlorine is in the pool at the beginning of the sixth week?

48. What happens to the amount of chlorine after an extended period of time?

TREE FARM **In Exercises 49 and 50, use the following information.**
Suppose a tree farm initially has 9000 trees. Each year 10% of the trees are harvested and 800 seedlings are planted.

49. Write a recursive rule for the number of trees on the tree farm at the beginning of the *n*th year. How many trees remain at the beginning of the fourth year?

50. What happens to the number of trees after an extended period of time?

FOCUS ON CAREERS

REAL LIFE **PHYSICIAN**
Physicians diagnose illnesses and administer treatment such as drugs, the focus of Exs. 51–54. In 1996 there were about 738,000 physicians in the United States.

INTERNET CAREER LINK
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DOSAGE **In Exercises 51–54, use the following information.**
A person repeatedly takes 20 milligrams of a prescribed drug every four hours. Suppose that 30% of the drug is removed from the bloodstream every four hours.

51. Write a recursive rule for the amount of the drug in the bloodstream after *n* doses.

52. What value does the drug level in the person's body approach after an extended period of time? This value is called the *maintenance level*.

53. Suppose the first dosage is doubled (to 40 milligrams), but the normal dosage is taken thereafter. Does the maintenance level from Exercise 52 change?

54. Suppose every dosage is doubled. Does the maintenance level double as well?

55. CRITICAL THINKING Give an example of a sequence in which each term after the third term is a function of the three terms preceding it. Write a recursive rule for the sequence and find the first 8 terms.

Test Preparation

56. MULTIPLE CHOICE What is the fifth term of the sequence whose first term is $a_1 = 10$ and whose nth term is $a_n = 2a_{n-1} + 9$?

(A) 67 (B) 143 (C) 286 (D) 295 (E) 599

57. MULTIPLE CHOICE What is a recursive equation for the sequence 4, −6.6, 10.89, −17.9685, . . .?

(A) $a_n = (-2.6)a_{n-1}$ (B) $a_n = (-1.65)a_{n-1}$

(C) $a_n = (2.6)a_{n-1}$ (D) $a_n = (1.65)a_{n-1}$

★ Challenge

58. PIECEWISE-DEFINED SEQUENCE You can define a sequence using a piecewise rule. The following is an example of a piecewise-defined sequence.

$$a_1 = 7,\ a_n = \begin{cases} \dfrac{a_{n-1}}{2}, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1, & \text{if } a_{n-1} \text{ is odd} \end{cases}$$

a. Write the first ten terms of the sequence.

b. LOGICAL REASONING Choose three different values for a_1 (other than $a_1 = 7$). For each value of a_1, find the first ten terms of the sequence. What conclusions can you make about the behavior of this sequence?

EXTRA CHALLENGE
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MIXED REVIEW

EVALUATING POWERS **Evaluate the power.** (Review 1.2 for 12.1)

59. 2^5 **60.** 6^4 **61.** 8^4 **62.** 12^3

63. 26^3 **64.** 10^5 **65.** 18^3 **66.** 3^7

OPERATIONS WITH RATIONAL EXPRESSIONS **Perform the indicated operation and simplify.** (Review 9.5)

67. $\frac{3}{5x} + \frac{3}{7x}$ **68.** $\frac{-2}{7x} - \frac{5}{3x}$ **69.** $\frac{x+1}{x^2-9} - \frac{5}{x-3}$

70. $\frac{2x^2}{3x+5} - \frac{14}{x+7}$ **71.** $\frac{4x+1}{x^2-4} - \frac{3}{x-2}$ **72.** $\frac{x^2-1}{x+2} - \frac{3}{x+1}$

FINDING POINTS OF INTERSECTION **Find the points of intersection, if any, of the graphs in the system.** (Review 10.7)

73. $x^2 + y^2 = 4$, $2x + y = -1$ **74.** $x^2 + y^2 = 25$, $y = x - 1$ **75.** $x^2 + 4y^2 = 16$, $y = 3x + 1$

76. $x^2 + y^2 = 10$, $4x + y = 6$ **77.** $x^2 + y^2 = 30$, $y = x + 2$ **78.** $16x^2 + y^2 = 32$, $\frac{1}{4}x - \frac{1}{2}y = 2$

WRITING TERMS **Write the first six terms of the sequence.** (Review 11.1)

79. $a_n = 8 - n$ **80.** $a_n = n^4$ **81.** $a_n = n^2 + 9$

82. $a_n = (n + 3)^2$ **83.** $a_n = \frac{n}{n+4}$ **84.** $a_n = \frac{n+3}{n+1}$

QUIZ 2

Self-Test for Lessons 11.4 and 11.5

Find the sum of the infinite geometric series if it has one. (Lesson 11.4)

1. $\sum_{n=0}^{\infty} 4\left(\frac{1}{9}\right)^n$ **2.** $\sum_{n=1}^{\infty} 5\left(-\frac{6}{7}\right)^{n-1}$ **3.** $\sum_{n=0}^{\infty} -\frac{3}{8}\left(\frac{4}{7}\right)^n$ **4.** $\sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{5}{4}\right)^n$

Find the common ratio of the infinite geometric series with the given sum and first term. (Lesson 11.4)

5. $S = 5, a_1 = 1$ **6.** $S = 12, a_1 = 1$ **7.** $S = 24, a_1 = 3$

Write the repeating decimal as a fraction. (Lesson 11.4)

8. 0.888 . . . **9.** 0.1515 . . . **10.** 126.126126 . . .

Write the first five terms of the sequence. (Lesson 11.5)

11. $a_1 = 5$, $a_n = a_{n-1} + 3$

12. $a_0 = 1$, $a_n = 4a_{n-1}$

13. $a_1 = 17$, $a_n = a_{n-1} + n$

14. $a_1 = 1, a_2 = 2$, $a_n = a_{n-1} - a_{n-2}$

15. $a_1 = 2, a_2 = 4$, $a_n = a_{n-1} \cdot a_{n-2}$

16. $a_1 = 10, a_2 = 10$, $a_n = a_{n-2} + a_{n-1}$

17. **BALL BOUNCE** You drop a ball from a height of 8 feet. Each time it hits the ground, it bounces 40% of its previous height. Find the total distance traveled by the ball. (Lesson 11.4)

MATH & History — The Fibonacci Sequence

APPLICATION LINK www.mcdougallittell.com

THEN

IN 1202 the mathematician Leonardo Fibonacci wrote *Liber Abaci* in which he proposed the following rabbit problem.

Begin with a pair of newborn rabbits that never die. When a pair of rabbits is two months old, it begins producing a new pair of rabbits each month.

Month	1	2	3	4	5	6	. . .
Pairs at start of month	1	1	2	3	5	8	. . .

This problem can be represented by a sequence, known as the Fibonacci sequence. The numbers that make up the sequence are called Fibonacci numbers. The ratio of two Fibonacci numbers approximates the same number, denoted by Φ. The Greeks called this number the golden ratio.

1. Draw a tree diagram to illustrate the sequence.
2. If the initial pair of rabbits produces their first pair of rabbits in January, how many pairs of rabbits will there be in December of that year? What happens to the rabbit population over time?

NOW

TODAY we know that Fibonacci numbers occur in nature, such as in the spiral patterns on the head of a sunflower or the surface of a pineapple.